Embedding of Space-Time

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Abstract

This article presents a new approach for the geometry of space-time with "Dark Matter" and "Dark-Energy" effects.
Motivated by an introduced merging of classical de-Sitter space (**dS**) and Anti-de-Sitter space (**AdS**) in a 6-D light-cone and the embedding of the Schwarzschild space (**Sch**) in a 6-D space, an embedding of stationary isotropic metrics into a common sphere in flat $R^{2,8}$ [1] is proposed. Integrability conditions for the embedding lead to inequalities between radial parameter and Schwarzschild radius, depending on the curvature radius of the global sphere. These inequalities define a minimal length structure in 3-D space. The minimal length corresponds to the size of a Kaluza-Klein (K-K) sphere (Section 2). A modification of the phase of the K-K sphere then leads to a non-stationary metric (section III) and diagonalization results in a non-linear PDE-system. We derive the associated Hamilton and Hamilton-Jakobi equations and we offer a particle model for the equations with the phase as the particle action. We motivate special initial values and explicitly integrate the characteristic equations. At initial time, the metric has no time dimension and an almost 2-dimensional spatial extension. In the model the gravitation "constant" and the speed of light become time and space dependent. The metric tends for large time at any spatial point to the Schwarzschild metric, but behaves quite different in the limit of large large spatial distances. This coordinate dependency leads to a redshift of radiation from far objects as also to additional attractive forces and so describes main effects concerned with "dark energy" and "dark matter".
In section IV properties of the complex space, arising from an orthogonal projection of the Kaluza-Klein dimension onto the light cone, are pointed out. In these coordinates any one of the embedded spaces becomes, in a coherent limit, a minimal Lagrangian submanifold, which is just a 2-D sphere. After this, in section V the ansatz is discussed under the point of the view of classical Kaluza-Klein theory [WL,Le,Str,Bl].

I Introduction

Classically, **dS** and **AdS** metrics belong to the Friedman-Lemaitre cosmological model. **DS** describes an absolute homogenous, expanding, and **AdS** a contracting universe [St,Dr,Ri,Bl]. Due to the observations of an expanding universe in the last century, **AdS** is rejected as a model of space-time, but **AdS** geometry plays an important role in modern physical theories [RO]. The classical **dS** and **AdS** are just hyper-spheres in a five dimensional flat space, but the signature of both flat metrics differ (see [St, Mos] and Appendix **A**). While the causal structure of **dS** is causal, the one of **AdS** is acausal and allows closed time-like paths [Mos].

The two (to my knowledge) known embeddings of **Sch** into a flat 6-dimensional space, the Kasner and the Fronsdal embedding [Mo], have similar but different signatures. Note that Kasner's flat metric is of **AdS** type, while Fronsdal's is of **dS** type. Fronsdal's embedding could be extended to the Kruskal metric [Mo, DP] and carries the same causal structure as **dS.** Kasner's embedding is acausal and not extendable. So Kasner's embedding is nowadays almost unvalued, but it will become the starting point of our common embedding, due to the observations below about the **dS/AdS** spaces. In this article I propose an embedding of **Sch** (together with a large set of isotropic, static

1 I use $K^{n,m}$ for an n+m dimensional space and K^n_m for an n-dimensional space, both with m time-like dimensions.

metrics) into a manifold **M**, where the embedding combines Kasner's ansatz with the causality of the de-Sitter space and **M** contains also **dS** and **AdS** as submanifolds.
The first aim is now to merge the embeddings of **AdS and dS** and to look for similarities to the embeddings of the Schwarzschild space.

Some Notations are in order:

g, ds^2 is used for a metric (line element)

G for the Einstein gravitation tensor, without any cosmological constant.

The expression "Newton potential U_N of a metric", should be understood in the following way: Writing the form factor of the time coordinate of a metric as $g_{00} := g_{tt} = 1 + U$ one gets in the non-relativistic limit, via geodesic equation, the Newton potential U_N as

$$\ddot{r} \approx -\Gamma^r_{tt} = \frac{1}{2} \cdot g^{rr} \cdot g_{tt,r} \approx -\frac{1}{2} \cdot g_{tt,r} \approx -U_{,r}/2 \Rightarrow U_N \approx U/2 \, .$$

If ($(x_i, i=0 \ldots n)$ are coordinates in $\boldsymbol{R}^{d,(n-d)}$, the first three spatial entries are used as the usual 3-d spatial coordinates: $\boldsymbol{r} = (x_d, x_{d+1}, x_{d+2}),\ r := |(\boldsymbol{r})|$. The scalar product is given by

$$\langle x, y \rangle = \sum_0^{d-1} x_i \cdot y_i - \sum_d^{n-1} x_i \cdot y_i$$

dS and AdS as conic sections and the Schwarzschild metric

(For definition and a set of elementary properties of classical **dS**/**AdS**, see Appendix A). The embedding of classical **AdS** and **dS** could be merged in the 6-dimensional flat space $\boldsymbol{R}^{2,4}$ as simple different sections of the same hyper sphere, the "light cone" **K**

$$\mathbf{K} = \{ x \in \boldsymbol{R}^{2,4} : x^2 := \langle x, x \rangle = 0 \} \, .$$

One gets **dS** or **AdS** as the sections $x_1^2 = 1$ or $x_5^2 = 1$, respectively (we use the indices 2,3,4 for the usual space dimensions). The subspaces could be transformed into each other via a simple rotation of the $x_1 \text{ and } x_5$ axis around one of the others (e.g. the x_0 -axis). The radial part of the stationary metrics of **dS** and **AdS** reflects this also: For this consider the cone $x^2 + r^2 = y^2$ and constant sections at α

dS: $$y = \alpha :\Rightarrow x^2 + r^2 = \alpha^2 \Rightarrow dr^2 + dx^2 = \frac{dr^2}{(1 - (r/\alpha)^2)}$$

AdS: $$x = \alpha :\Rightarrow \alpha^2 + r^2 = y^2 \Rightarrow dr^2 - dy^2 = \frac{dr^2}{(1 + (r/\alpha)^2)}$$

As is pointed out in Appendix B, using equivalent parametrization of **K** leads to qualitatively completely different kinds of metrics for **dS** and **AdS.** Circular coordinates induce a Friedman-Lemaitre-Robertson-Walker (FLRW) metric on **dS,** but a stationary metric on **AdS.** Hyperbolic coordinates, on the other hand, induce the opposite kind of metrics, a stationary on **dS** and a FLRW metric on **AdS**. This is an example, that the metric property, to be stationary or not, is neither a characteristic attribute of the geometrical form of the manifold nor of the type of parametrization, it depends on both.

So the restriction of a global coordinate system of the embedding space may lead to a stationery metric on one submanifold and an time-dependent on the other one, but this features may interchange with another global parametrization. So we can't expect to find a common parametrization of the searched embedding space **M** and certain restrictions to get known metrics of **dS, AdS** and **Sch.**
As already mentioned, for **Sch** there exists also an embedding into $R^{2,4}$, the Kasner embedding, while the embedding space of Fronsdal is $R^{1,5}$ [Mo]. The geometry of both embeddings is difficult and without any further similarities to **dS** and **AdS,** in particular it is not a submanifold of **K** and is not projective, while **dS** and **AdS** are projective spaces**.**

But there is another similarity between the Schwarzschild metric and the stationary metrics of **dS** and **AdS.** The radial part of the Schwarzschild metric $-g_{rr}=(1-r_0/r)^{-1}$ comes from a conic section:

$$x^2=4\mathrm{r}_0\cdot(r-r_0) \Rightarrow dr^2+dy^2=\frac{dr^2}{(1-(r_0/r))}$$

This conic section arises from $x^2+r^2=y^2$ through a rotation around the x-axis with angle $\pi/4$, followed by a translation $r\to r-r_0$ and finally cut at $y=2\cdot r_0$

$$y\to(y-r)/\sqrt{(2)},\, r\to(r+y)/\sqrt{(2)} \quad\Rightarrow\quad x^2=(y-r)(y+r)\to x^2=2\,y\cdot r\to x^2=4\mathrm{r}_0(r-r_0)$$

Comparing the parameters of the cutting planes $y=2\cdot r_0$ and $y=\alpha$ leads to

$$\boldsymbol{r_0=\alpha/2} \tag{I.1}$$

A relation of this kind, between r_0 and α, we will get later again.

II Embedding in a common Space

Due to the considerations in the last section it seems appropriate to restrict **M** to be a hypersphere in $\boldsymbol{R}^{2,k}, k>4$. Using k=5, i.e. seven dimensions, does not lead to essentially better results, moreover it is hard to understand why two independent extra space-like dimensions and only one time-like dimension should exist. For several reasons I also think that the number of dimensions must be even such that a complex structure could be defined on it. For k=6, i.e. eight dimensions, one get the product of two Minkowski spaces (or a complexified) . Without any further demands it would be possible to embed all three spaces (**dS, AdS** and **Sch)** in the flat $\boldsymbol{R}^{2,6}$. But for **dS/AdS** we found a light-cone as embedding space and we want to keep this characteristic and we write it as

$$K=\{(x,y):x\in R^{1,3}, y\in R^{1,3} \text{ and } 0=\langle x,x\rangle+\langle y,y\rangle=x_0^2+y_0^2-r_x^2-r_y^2\}.$$
$$r^2:=r_x^2+r_y^2,\quad r_x^2:=\sum_{j=1}^{3}x_i^2,\quad r_y^2:=\sum_{j=1}^{3}y_i^2 \tag{II.1}$$

Choosing spatial coordinates $y_i, i=1\ldots3$ on a straight line, restricts this cone to the light cone in $\boldsymbol{R}^{2,4}$, which we used to embed **dS/AdS**. For the Schwarzschild space we not found neither a similar simple section nor an embedding symmetric due to the two Minkowski subspaces. As such symmetry reasons, we demand now for the embedding of **Sch**, that the angles between the two sets of space dimensions should match. But then we loose again two degrees of freedom and receive a cone in $\boldsymbol{R}^{2,3}$ and so a 5-dimensional manifold which not could embed a Schwarzschild submanifold. So we extend the space again and end up with the 10 dimensional space $\boldsymbol{R}^{2,8}$. The space is now the product of two classical Kaluza-Klein spaces. We have the decomposition

$$\boldsymbol{R}^{2,8} = \boldsymbol{R}^{1,4} \circ \boldsymbol{R}^{1,4}$$

and start with the ansatz,

$$M = \{(x, y) : x \in R^{1,4}, y \in R^{1,4} \text{ and } \langle x, x \rangle + \langle y, y \rangle = -\alpha^2\}. \qquad \text{(II.2)}$$

Kaluza-Klein theory postulates a S^1 submanifold so we conclude with II.1

$$\rho^2 := x_4^2 + y_4^2, \quad -\alpha^2 = \langle x, x \rangle + \langle y, y \rangle = x_0^2 + y_0^2 - \rho^2 - r^2 = -\rho^2 \qquad \text{(II.3)}$$

So the diameter of the Kaluza-Klein sphere is constant and equal to the size of the space. In the following we use normalized coordinates, i.e. a curvature radius $\alpha = 1 \Rightarrow \rho = 1$. In a stationary, isotropic metric this means that time and radial parameters are measured in multiples of α.
The restriction that all angles between the corresponding x- and y-space dimensions should be equal we express via

$$x_i = r_i \cos(\phi), \, y_i = r_i \sin(\phi) \text{ for } i = 1,2,3 \quad .$$

Using spherical coordinates for the space dimensions and $x_4 = \rho \cos(\beta), \, y_4 = \rho \sin(\beta)$ for the Kaluza-Klein, the metric becomes

$$\begin{aligned} ds^2 &= dx_0^2 + dy_0^2 - dr^2 - r^2 d\phi^2 - r^2 d\Omega - d\rho^2 - \rho^2 d\beta^2 \\ &= dx_0^2 + dy_0^2 - dr^2 - r^2 d\phi^2 - r^2 d\Omega - d\beta^2 \end{aligned} \quad . \qquad \text{(II.4)}$$

To obtain an embedding for the Schwarzschild space we set

$$x_0 = r \cos(\omega), \, y_0 = r \sin(\omega) \quad \text{and get}$$

$$ds^2 = r^2 d\omega^2 - d\beta^2 - r^2 d\phi^2 - r^2 d\Omega \qquad \text{(II.5)}$$

Let's now choose the following embedding

$$\omega = \kappa \sinh(t), \, \phi = \kappa \cosh(t), \, \kappa = a/r$$

and receive

$$\begin{aligned} ds^2 &= a^2 dt^2 - r^2 d\kappa^2 - d\beta^2 - r^2 d\Omega \\ &= a^2 dt^2 - f^2 dr^2 - d\beta^2 - r^2 d\Omega, \quad f := r \frac{d}{dr} \frac{a}{r} = \frac{da}{dr} - \frac{a}{r} \end{aligned} \qquad \text{(II.6)}$$

Summarized the complete parametrization is

$$\begin{aligned} &x_0 = r \cos(\kappa \sinh(t)), && y_0 = r \sin(\kappa \sinh(t)) \\ &x_i = r \cos(\kappa \cosh(t)) \cdot \sigma_i, && y_i = r \sin(\kappa \cosh(t)) \cdot \sigma_i \text{ for } i = 1,2,3 \\ &x_4 = \cos(\beta), && y_4 = \sin(\beta) \end{aligned} \qquad \text{(II.7)}$$

where σ_i are the usual spherical unit coordinates (and $\kappa = a/r$).
Comparing the metric above with a "target" metric of the general kind

$$ds^2 = a^2(r)dt^2 - b^{-2}(r)dr^2 - r^2 d\Omega \quad , \tag{II.8}$$

leads the equation

$$\beta'^2 = b^{-2} - f^2 \equiv b^{-2} - (a' - a/r)^2 =: m^2, \quad (a' := da/dr) \quad . \tag{II.9}$$

It has a solution for $b^2 \geq 0$ if also $m^2 \geq 0$ or

$$1 \geq (b \cdot (a' - a/r))^2. \tag{II.10}$$

This embedding provides a wide class of stationary metrics, including Schwarzschild - De-Sitter metrics, that are metrics with

$$\begin{aligned} &-g_{rr}^{-1} = b^2 = g_{tt} = a^2 = 1 + V = -r_0/r + \lambda r^2 \\ &\Rightarrow \kappa^2 = (1 - r_0/r)/r^2 + \lambda, \quad b \cdot (a' - \frac{a}{r}) = \frac{V'}{2} - \frac{(1+V)}{r} = \frac{1}{r}(\frac{3r_o}{2r} - 1) \end{aligned} \quad . \tag{II.11}$$

Here the additional curvature constant λ appears, but has no influence on II.10 (and not on II.12 below) , as we see with the help of the last formula.. But a second, arbitrary curvature constant (we have already the global α) needs to be explained again in some way and so we set just $\lambda = 0$.
The relation II.10 now simplifies to

$$r \geq \left| (\frac{3}{2} \cdot \frac{r_0}{r} - 1) \right| \quad . \tag{II.12}$$

Setting $r_0 = 0$ one gets the first result $r \geq 1$, that is: Even for the flat target space the considered embedding has a gap at the origin of the coordinate system. So the embedding implies a natural minimal structure of the space with size 1, that is the size of the global curvature α .

<u>So we assume that $\boldsymbol{\alpha}$ is of the size of a Planck length !</u>

<u>We will not need and use, how small α exactly is and it is enough to know, that it is very small at any macroscopic scale. So even if we would assume $\alpha = 10^{20}$, the radius $r = 1$ would correspond just to the diameter of a proton and would be highly accurate negligible compared to terrestrial or larger distances.</u>
We claim that II.12 should hold for all values of r , where also $a^2 \geq 0$ holds and obtain, inserting $r = r_0$ in II.12, $r_0 \geq 1/2$ as a necessary condition. Further, simple dividing and subtracting the relation $r \geq r_0 \geq 1/2$ results in

$1/2 \geq \frac{3}{2} \cdot \frac{r_0}{r} - 1 \geq \frac{3}{4\mathrm{r}} - 1 = -r + \frac{(r - 1/2)^2 + 1/2}{r} \geq -r$ and with $r \geq r_0 \geq 1/2$ in

$r \geq \frac{3}{2} \cdot \frac{r_0}{r} - 1 \geq -r$ which is II.12 . Only for $\boldsymbol{r_0 \geq 1/2}$ could the Schwarzschild space **Sch** be entirely included in this model. Again we found the value $r_0 = 1/2$ as already in II.4. But for $r_0/r \leq 1$ the right hand side of II.12 is always less than one and so the inequality is fulfilled if

$$r \geq r_0 \geq 1/2 \quad \text{or} \quad r \geq \max(1, r_0) \quad . \tag{II.13}$$

In the Newtonian limit one derives $r_0 = 2 \cdot mass$ (see e.g. [St]) . Applying this to the inequality $r_0 \geq 1/2$ says, that our model keeps valid for masses of size larger than the Planck mass . This is small enough for any macroscopic considerations.

For $a^2 = b^2 = 1+V$ one has

$$\beta'^2 = m^2 \equiv \frac{1}{a^2} - f^2 = \frac{1}{a^2}(1-f_a^2)$$
$$f_a := a \cdot f = (\frac{-1}{r} + (\frac{1}{2}\frac{d}{dr} - \frac{1}{r})V) = -\frac{1}{r}(1+V-\frac{r}{2}\frac{dV}{dr}) \tag{II.14}$$

and so, for the Schwarzschild – de-Sitter metrics, the phase β of Kaluza-Klein's S^1 is given by the integral (for completeness we also keep λ here)

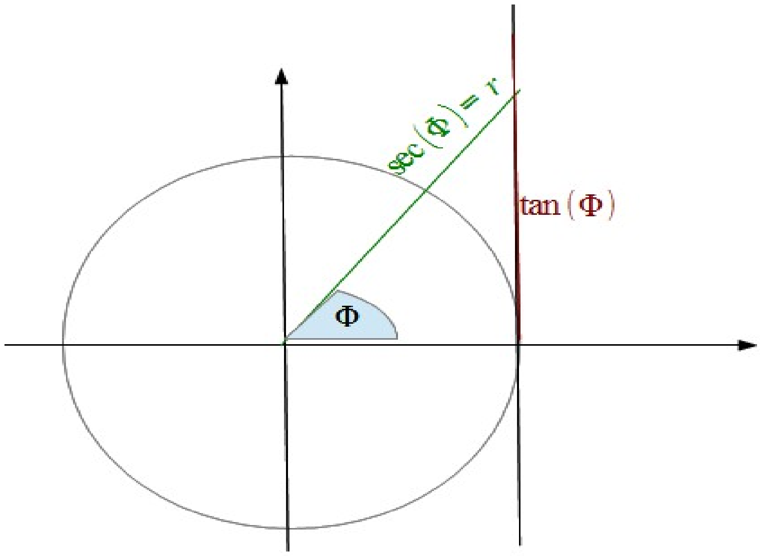
$$\beta = \pm \int \frac{dr}{r} \sqrt{\frac{r^2 - (1-3\mathrm{r}_0/2\mathrm{r})^2}{1-r_0/r+\lambda r^2}} \tag{II.15}$$

As for the Kasner and Fronsdal embedding, this last primitive could not be explicitly presented, but it behaves well for all values, where the real square root in the integrand is defined.
Extending to a complex space, β is defined everywhere. It just becomes imaginary and so the radius of the Kaluza-Klein dimension now decreases/increases (depending on the sign in the last formula). We may also use this embedding to get **dS** and **AdS.** For those we get , with $a^2 = 1 \mp r^2 \Rightarrow f^2 = (ar)^{-2} \Rightarrow m^2 = a^{-2}(1-r^{-2})$, a negative m^2 for $r<1$. So for **AdS** we have the same critical distance $r=1$ as for **Schw.** But for **dS** the expression $m^2 = -r^{-2} < 0$ is everywhere negative and so the phase $\beta = \pm i \ln(r)$ complex, which leads to a non constant radius of the KK-dimension $\rho = r^{\pm 1}$.
For $r_0 \geq 1/2$ the sign of m^2 is positive for all $r > r_0$. Beyond r_0 it first turns negative, but because $a^2 < 0$ for $0 \leq r < r_0$ and $\lim_{r \to 0} f_a^2 = \infty$ the sign of m^2 becomes again positive near the origin at $r=0$. The region of negative m^2 (or regions) form a shell (or shells) around the origin.
If $x_\mu(t_2) = x_\mu(t_1)$ and $y_\mu(t_2) = y_\mu(t_1)$ for constant $r \geq r_0$ and $\kappa^2 = (1-r_0/r)/r^2 \neq 0$, we have $\sinh(t_2) = \sinh(t_1) + 2 l \pi/\kappa$ and $\cosh(t_2) = \cosh(t_1) + 2 n \pi/\kappa$ with $l, n \in \mathbb{N}$. This induces $t_1 = t_2,\ l = n$ and therefore, with increasing parameter time, the path on ***M*** never reaches the same point again. The embedding is causal in this sense.
The phase factor κ has a maximum at $r = 3/2 \cdot r_0$ and $\kappa(r_0) = \kappa(\infty) = 0$ and so $\forall r_a : r_0 < r_a < 3/2 \mathrm{r}_0\ \exists r_b > 3/2 \mathrm{r}_0 : \kappa(r_b) = \kappa(r_a)$. The complex coordinates $u_\mu = x_\mu + i y_\mu$ on the light cone subspace ($(x_\mu, y_\mu), \mu = 0 \ldots 3$) of **M** at (t, r_a) and (t, r_b) have the same phase and only differ in their "amplitude". Further, on a sphere $r = \mathrm{constant}$, with increasing parameter time, the time and space dimensions oscillate, like ordinary waves. Not so the Kaluza-Klein dimensions, which only depends on r . At the event horizon ($\kappa = 0$), the oscillation is frozen. Passing through this singularity, the oscillation starts again, but in the opposite sense. The time and radial part of the 4 -D metric changes sign and the radius of the Kaluza-Klein S^1 – sphere decreases with decreasing r . At $r=0$, all dimensions, without the Kaluza-Klein one (the whole space-time), shrink to zero size.
The phase β in this section is selected in such a way that the initial metric becomes the desired target space. But without this, there is no reason that β only depends on the radial coordinate. In the next section we will examine a modified phase β .
We finally give another interpretation of equation II.9 for the flat Schwarzschild space. For this we write the equation as

$d\beta^2 + (\frac{dr}{r})^2 = dr^2$, which results with $r = \cosh y$ and $d\Phi := \frac{dy}{\cosh(y)}$ in

$$d\beta^2 + dy^2 = dr^2 + d\Phi^2 \quad . \qquad \text{(II.16)}$$

The function $\Phi(\xi) = gd(\xi) = \int_0^\xi \frac{dy}{\cosh(y)} = \text{atan}(\sinh(\xi))$, "which arises in the inverse equations for the Mercator projection, and which expresses the latitude $\Phi(\xi) = gd(\xi)$ in terms of the vertical position ξ on the Mercator map" (Wolfram.com).
Since $\cosh(\xi) = \sec(\Phi) := 1/\cos(\Phi)$ we may write also $r = \sec(\Phi)$.



The right hand side of II.16 just describes some unusual kind of latitudinal distance measure on the globe, and the left hand side the corresponding length preserving measure on the Mercator map. For small latitudes with $\Phi \approx \tan\Phi$ we have the relation $\Phi^2 + r^2 \approx 1$ and with $d\beta^2 = (1 - f^2) dr^2 = \sin^2(\Phi) dr^2$, we see that $d\beta$ measures the radial distance, weighted with the sine of the latitude $d\beta = \sin(\Phi) dr$.

III Extended Schwarzschild Metrics

III.1 Preliminary remarks

Before considering modified manifolds, which arise from the change of the phase β in the last section, we review some general facts. For this we first review the concept of the "proper length" or "proper distance". In special relativity the "proper length" is the length with respect to the rest frame of the object, which is measured. Consequently the proper distance between two objects has to be measured in a frame, where both objects are at rest, regardless if they have a relative movement to each other or not. This is the frame of a static or stationary observer. So for example, the proper distance sun-earth is one astronomical unit, even if a satellite traveling from earth to the sun measures a different distance. For a general static (radially symmetric) metric

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 - r^2 d\Omega \qquad \text{(III.1.1)}$$

the proper distance l is given as $l: dl^2 = -g_{rr} dr^2$ [MT,G] and is the negative arc length $-ds^2$ of the space-like distance at synchronous time ($dt/ds=0$). Besides the proper radial distance we have for the radial velocity as a static observer measures,

$$\mathrm{v}_S^2 = g^{tt}\left(\frac{dl}{dt}\right)^2, \quad g^{tt} := g_{tt}^{-1} \qquad \text{(III.1.2)}$$

[MT,Oi]. Using these relations, a straight line element($d\Omega=0$) is just

$$ds^2 = g_{tt}\left(1-\mathrm{v}_S^2\right)dt^2 \quad . \qquad \text{(III.1.3)}$$

If g_{tt} is constant (let use 1 without loss of generality) , this is just the line element of special relativity. Moreover for light-like geodesics we get $dt^2 = -g_{rr} dr^2 = dl^2$ and so $\mathrm{v}_S^2 = (dl/dt)^2 = 1$. So for the static observer, who measures time and length, anything looks just like special relativity, despite that the space is not flat. We will call this type of metrics and spaces **"almost flat" (AF)**. But of course there are differences to a flat space; for example measuring the circumference of a circle centered at $r=l=0$, which is $2\mathrm{r}\pi$ and not $2\mathrm{l}(r)\pi$. Another difference is that the Einstein tensor, and so via Einstein equation the energy – momentum tensor, generally does not vanish.

With the results in [Dr 8.31 ff] and $-g_{rr} := Q^{2,} \; g_{tt}=1$ one arrives at

$$G_0^0 = \frac{1}{r^2}\frac{d}{dr}\left(r\left(Q^{-2}-1\right)\right), \;\; G_r^r = \frac{\left(Q^{-2}-1\right)}{r^2}, \;\; G_2^2 = G_3^3 = \frac{1}{2\mathrm{r}}\frac{d}{dr}\left(Q^{-2}\right) \quad . \qquad \text{(III.1.4)}$$

From this immediately follows $G_0^0 = \sum_{j=1}^{3} G_j^j$, which is, if all components are positive or zero the strong energy condition [G][2]

$$G_0^0 \ge 0 \;\wedge\; G_0^0 \ge G_j^j \;\; \forall j=1,2,3 \;\wedge\; G_0^0 \ge \sum_{j=1}^{3} G_j^j \quad ,$$

which is interpreted as an indication for "ordinary matter" [G,MT] . Conversely, if all components are zero or negative, we get $G_0^0 \le 0, \; G_0^0 \le G_j^j \;\; \forall j=1,2,3$ and so G lacks any energy-condition. A metric, which looks like a **SR**-metric, usually is a vacuum, so it should not look like a metric with matter. In a vacuum there is nothing we can identify with positive energy. So we only accept **AF** metrics, if the Einstein tensor does <u>NOT</u> fulfills any energy-condition. We call these last inequalities

$$G_0^0 \le 0 \;\wedge\; G_0^0 \le G_j^j \;\; \forall j=1,2,3 \;\wedge\; G_0^0 \le \sum_{j=1}^{3} G_j^j$$

the "**vacuum energy condition**". Further we expect that asymptotically the vacuum should become a vacuum in the normal sense, that is G vanishes at infinity, which also implies that the space is asymptotically Ricci-flat ($G=0 \Rightarrow tr(G)=0 \Rightarrow tr(Ric)=0$). This condition is weaker than the common "asymptotical flatness", which means that the metric tends to a Minkowski metric at spatial infinity.
Let's make the following definition:

Definition:

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2) Note [G] uses upper lower indices with respect to the Minkowski metric η_{ij} and so his energy-momentum tensor is $T_{ij} = \eta_{il} G_j^l$!

A **vacuum** is a domain in space-time, where the metric is **almost flat** and the **vacuum energy condition** holds with $\lim_{r\to\infty} G(r)=0$. In a stronger form of the definition, we demand additionally $G\leq 0,$ i.e. $G_j^j\leq 0 \quad \forall\, j$.
Sometimes space-time domains, which fulfill no energy condition, are called "exotic matter". We do not use this terminology and/or interpretation in the following.

We now return to the geodesic equations. With $g_{tt}=b^2(r),\ -g_{rr}=Q^2(r)/b^2(r)$ **,** the Lagrange function associated with the metric III.1.1 is

$$L = b^2\dot{t}^2-(Q/b)^2\dot{r}^2-r^2(\dot{\theta}^2+\sin^2(\theta)\dot{\phi}^2) \quad . \tag{III.1.5}$$

The dot (e.g. in $\dot{r}$) denotes the derivative with respect to an affine parameter σ along the geodesic, where for time-like geodesics σ could be identified with the arc length (or the eigentime). So from $ds^2 = (g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)d\sigma^2$, $\mathcal{L}=g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ one gets for time-like geodesics, using $\sigma=s$, $\mathcal{L} = const. = 1$ and with $ds^2=0$ for light-like (null-geodesics) $\mathcal{L} = 0$ [St]. The geodesic equations for the angular and the time parameter have the known standard form and solutions $\theta=\pi/2,\ r^2\dot{\phi} = M = const.$ and $b^2\dot{t} = A = const$ and one gets the equation

$$\frac{A^2}{b^2} - Q^2\frac{\dot{r}^2}{b^2} - \frac{M^2}{r^2} = \mathcal{L}\ ,\ \ \mathcal{L}=0{,}1.\ \ A^2\geq 1 \quad . \qquad \text{(III.1.6}$$

This equation is a necessary condition for the radial Euler-Lagrange equation to hold and for $\dot{r}\neq 0$ it is also sufficient (proven by differentiating with respect to the parameter along the curve). So for $\dot{r}\neq 0$ it can be used instead of the radial Euler-Lagrange equation. For radial geodesics through the origin ($M=0$) one gets $Q^2\dot{r}^2=A^2-\mathcal{L}\,b^2$.
Inserting this equation into III.1.2 one obtains

$$\mathrm{v}_S^2=b^{-2}\left(\frac{dl}{dt}\right)^2=\frac{1}{b^2}\left(\frac{Q}{b}\frac{dr}{d\sigma}\frac{d\sigma}{dt}\right)^2=Q^2\frac{\dot{r}^2}{A^2} = 1-\mathcal{L}\frac{b^2}{A^2} \quad . \tag{III.1.7}$$

For light-like geodesics ($\mathcal{L} = 0$) one again arrives at $\mathrm{v}_S^2=1$, but now for any function $b^2(r)$, and we can use $\sigma=t$ and therefore $A=1$. For time-like geodesics ($\mathcal{L}=1$), the last equation defines A via $A^2=b^2(1-\mathrm{v}_S^2)^{-1}$. Inserting this again in III.1.6 with $M=0$ leads to $Q^2\dot{r}^2 = b^2(1-\mathrm{v}_S^2)^{-1}\mathrm{v}_S^2$ and so summarized:

$$Q^2\dot{r}^2 = \frac{b^2\mathrm{v}_S^2}{(1-\mathrm{v}_S^2)} = \frac{1}{(1-\mathrm{v}_S^2)}\left(\frac{dl}{dt}\right)^2,\ \ Q^2\left(\frac{dr}{dt}\right)^2 = b^4\mathrm{v}_S^2 = b^2\mathrm{v}_S^2\left(\frac{dl}{dt}\right)^2 \quad . \tag{III.1.8}$$

If $\lim_{r\to\infty} Q(r)=\text{constant}$ and $\lim_{r\to\infty} b(r)=\text{constant}$ we determine A^2 in III.1.7, by defining initial conditions $\dot{r}=0,\text{for } r\to\infty$, and have for time-like geodesics

$$A^2=\lim_{r\to\infty} b^2(r) \tag{III.1.9}$$

(for light-like geodesics this initial condition makes no sense, we have always $A^2=1$).

Next we look at circular geodesics, that is at geodesics with $\dot{r}=0$. From III.1.5 one gets for the resulting radial Euler-Lagrange $\partial L/\partial dr=0$ equation $\frac{A^2}{b^4}\frac{db^2}{dr}=2\frac{M^2}{r^3}$ and after applying III.1.6

$$(\mathcal{L}+\frac{M^2}{r^2})\frac{db^2}{dr}=2b^2\frac{M^2}{r^3} \quad . \qquad \text{(III.1.10)}$$

The equation immediately shows that for **AF** spaces ($b^2=1$) no circular geodesics exists (M has to vanish), as they don't exist for usual flat spaces . We now look at a metric with $b^2=1-r_h/r\,,\ r_h\geq 0=\text{constant}$, which for $r_h=0$ is a **vacuum** as defined above. Any metric of this form is for $r_h=0$ an **AF** metric, but not necessarily a **vacuum**. The parameter r_h is the event horizon of the metric in the same meaning as for standard Schwarzschild spaces. For $r\rightarrow r_h$ time dilation dt/ds becomes infinite and behind it one has $\dot{t}<0$ ("time moves backward") and the metric changes its signature. For such kind of metrics, the limit value in III.1.9 is $A^2=1$ and III.1.10 becomes

$$(\mathcal{L}\frac{r^2}{M^2}+1)\frac{r_h}{r}=2(1-\frac{r_h}{r}) \overset{\mathcal{L}=0}{\Rightarrow} \frac{r_h}{r}=\frac{2}{3} \Rightarrow b^2=\frac{1}{3} \quad . \qquad \text{(III.1.11)}$$

Inserting this in (III.1.6 we get for circular null - geodesics ($\mathcal{L}=0 \rightarrow A=1$)

$$M=\frac{r}{b}=\frac{3}{2}\sqrt{3}r_h \;,\; \frac{d\phi}{dt}=\frac{M}{r^2}=\frac{2}{\sqrt{3}r_h}\,,\, E_M:=\frac{M^2}{r^2}=3 \qquad \text{(III.1.12)}$$

and we have with III.1.7 and $A^2=1$ for straight timelike geodesics, passing trough the origin ($M=0$) with the boundary condition $\dot{r}=0,\text{for } r\rightarrow\infty$

$$\mathrm{v}_S^2=b^{-2}(\frac{dl}{dt})^2=(Q\dot{r})^2=\frac{r_h}{r} \quad (\ \mathrm{v}_S^2=(Q\dot{r})^2=1 \text{ for light-like}). \qquad \text{(III.1.13)}$$

Inside the event horizon $0<r<r_h \Leftrightarrow b^2<0$ we have $\mathrm{v}_S^2>1$ and dl/dt becomes imaginary. At $r=r_h$, dl/dt is zero, whereas $dl/dr=Q^2/b^2$ goes to infinity, if $Q^2(r=r_h)\neq 0$. The "concept" of proper length lacks inside the event horizon. For this, the area $r<r_h$ is a gap or a single point.

We now consider the Newton limit ($r_h/r\ll 1$) of our equations for time-like curves. For III.1.11 we receive the familiar result $r_h/2=M^2/r$ for an orbit around a mass of $r_h/2$ and anything looks as usual. But the constant M^2 is undetermined. Differentiating of Equation III.1.13, the straight geodesics through the origin, leads to

$$\begin{aligned}\frac{d\,\mathrm{v}_S}{dt}&=\frac{d^2l}{dt^2}=Q\frac{d^2r}{dt^2}+\frac{dQ}{dr}\frac{\mathrm{v}_S^2}{Q^2}=-\frac{r_h}{2\mathrm{Q}}\cdot\frac{1}{r^2}=-\frac{\mathrm{v}_S^2}{2\mathrm{Q}}\cdot\frac{1}{r}\\ Q\frac{d^2r}{dt^2}&=-\frac{r_h}{2\mathrm{Qr}^2}(1+r\frac{d\log(Q^2)}{dr})\end{aligned} \quad . \qquad \text{(III.1.14)}$$

If we assume that $|\mathrm{v}_S|\ll 1$ and $r|d\log(Q)/dr|\ll 1$ and so also $l\approx Q\,r$, we get

$$\frac{d^2 l}{dt^2} \approx -Q\frac{r_h}{2}\cdot\frac{1}{l^2}\ ,\ \ \frac{d^2 r}{dt^2} \approx -\frac{r_h}{2Q^2}\cdot\frac{1}{r^2} \quad . \tag{III.1.15}$$

We have now different mass factors i.e Newton equations as for a mass of $Q_\infty^{-2} r_h/2$ or $Q_\infty r_h/2$, respectively and both factors differ from the Newtonian factor $r_h/2$. This is a consequence of the nature of g_{rr} and hence of the **vacuum.** $-g_{rr}$ does not become one in the Newton limit (of large r) , and consequently the proper length $dl = Q\,dr.$ will not match with the parameter length. So even for the velocities in the vacuum we get in the newton limit $\mathrm{v}_S = dl/dt \neq dr/dt$.
The concept of the proper velocity carries some practical difficulties. An observer on the earth measures the distance to objects in its galactic environment in terms of the proper length, but it is a geodesic observer and not a static one. If he looks at extra-galactic objects, he may be concerned as static, but now he is also outside of the gravitational effects of this objects and not be able to determine the proper distances between them. In both cases, he can't determine the velocity v_S . The difference in the forces may be seen as consequence of the different distances in the two coordinate systems.

As for the usual Schwarzschild metric, we can define a coordinate transformation into the coordinate system of a geodesic observer [St] and receive an associated LeMaitre-metric

$$\begin{aligned} dT &= dt + \sqrt{\frac{r_h}{r}}\frac{Q^2}{b^2}dr\,,\quad dR = dT + \sqrt{\frac{r}{r_h}}dr = dt + \sqrt{\frac{r}{r_h}}\frac{Q^2}{b^2}dr \\ ds^2 &= dT^2 - \frac{r_h}{r}dR^2 - r^2\,d\,\Omega \end{aligned} \tag{III.1.16}$$

The differential equations have the solutions

$$T = t + \int\sqrt{\frac{r_h}{r}}\frac{Q^2}{b^2}dr\,,\quad r = \left((R-T)\frac{3}{2}\sqrt{r_h}\right)^{2/3} \quad .$$

In this coordinate system, the metric is exactly the same as in the common theory. The difference to the Schwarzschild space is absorbed in the time parameter T .

III.2 Non stationary metrics.

We start again with metric II.6 but rename $t \rightarrow \tau$,

$$ds^2 = a^2\,d\,\tau^2 - f^2 dr^2 - d\,\beta^2 - r^2\,d\,\Omega \quad , \tag{III.2.1}$$

and consider a general phase $\beta(\tau, r)$ and a coordinate transformation $(\tau, r) \rightarrow (t, r)$ such that the metric is diagonal. We not transform the radial coordinate, so the surface term $r^2\,d\,\Omega$ remains unchanged and the metric keeps comparable to common radial symmetric metrics. As this reason the surface term plays no role in the following calculations and we omit witting it. Further we will restrict us in the following to the standard Schwarzschild scale factor $a^2 = 1 - r_0/r =: 1 - x$. Now with

$$d\beta = \omega\, d\tau - m\, dr \;=\; E dt - P\, dr\,,$$
$$\omega := \partial\beta/\partial\tau\,, \qquad m := -\partial\beta/\partial r\,, \qquad \text{(III.2.2)}$$
$$E := \partial(\beta\circ\tau)/\partial t = \omega\,\tau_{,t}\,,\;\; P := -\partial(\beta\circ\tau)/\partial r = -\omega\,\tau_{,r} + m$$

we receive the metric and the conditions

$$ds^2 = (a^2\tau_{,t}^2 - E^2)\,dt^2 - (P^2 + f^2 - a^2\tau_{,r}^2)\,dr^2 =: b^2 dt^2 - q^2 dr^2$$
$$a^2\tau_{,t}\tau_{,r} = -PE\,,\;\; E_{,r} = -P_{,t}\,,\;\; \tau_{,r,t} = \tau_{,t,r} \quad . \qquad \text{(III.2.3)}$$

For $E \equiv 0 \Leftrightarrow \omega \equiv 0$ this is just the same task as we already considered in section II and so we assume that E does not identically vanish. First also assume, that E depends only from t . With the ansatz $\tau_{,t} = \alpha E\,,\;\; \alpha = \text{const} \Rightarrow \omega\alpha = 1$. τ now separates into a sum of type $\tau = u(t) + v(r)$ and the new metric scale factors become

$$b^2 = (a^2\alpha^2 - 1)E^2\,,\;\; q^2 = P^2 + f^2 - a^2\tau_{,r}^2 = P^2(1 - \alpha^{-2}a^{-2}) + f^2 \quad .$$

Now E^2 can (and have to) be absorbed in the time coordinate $E\,dt \to dt$ by a simple integration or expressed in this final time parameter just as $E^2 = 1 = \beta_{,t}^2$. With $\alpha^2 = 2$ we receive $b^2 = (1 - 2r_0/r)$, $q^2 = P^2(1 - \omega^2/a^2) + f^2$, where P could be an arbitrary function of r since $0 = E_{,r} = -P_{,r}$. Using

$$P^2 = (b^{-2} - f^2)/(1 - \omega^2/a^2) \qquad \text{(III.2.4)}$$

we end up with the Schwarzschild metric

$$ds^2 = b^2 dt^2 - 1/b^2\, dr^{2,}\;\; b^2 := 1 - 2\mathrm{r}_0/r \quad . \qquad \text{(III.2.5}$$

Combined with the case $E = 0$ we have the metric factor

$$b^2 = 1 - r_0/r(1 + E^2)\,,\;\; E^2 = 0{,}1\,,\;\; \omega^2 = 0, 1/\sqrt{2} \quad . \qquad \text{(III.2.6)}$$

Together and a bit more generally, we used the transformation

$$\tau = t\sqrt{(1+E^2)} + v(r)\,,\;\; \beta = Et - h(r) = \omega\,\tau - (h(r) + \omega\, v(r)) \quad \text{with}$$

$$\omega^2 = \frac{E^2}{1+E^2} \Leftrightarrow E^2 = \frac{\omega^2}{1-\omega^2}\,,\;\; \tau_{,t}^2 = 1 + E^2 = \frac{1}{1-\omega^2}\,,\;\; h' = P\,, v' = -\omega\, P/a^2 \quad . \qquad \text{(III.2.7)}$$

That h and β becomes real we need $P^2 \geq 0$. Writing

$$P^2 = \frac{a^2}{b^4}\frac{A^2}{1-\omega^2}\,,\;\; A^2 := 1 - b^2 f^2 = 1 - \frac{b^2}{r^2 a^2}\left(\frac{3}{2}x - 1\right)^2 \qquad \text{(III.2.8)}$$

leads to the inequalities $b^2 \geq 0$ and $A^2 \geq 0$. We extend the first inequality, $b^2 \geq 0$, to an analogous of II.13

$$r \geq 2\mathrm{r}_0 \geq 1 \quad . \qquad \text{(III.2.9)}$$

Since in this region $b^2 \le a^2$ and $(3/2x-1)^2 \le 1$ holds, $A^2 = 1 - b_\infty^2/a^2(3/2x-1)^2/r^2$ is also indeed always positive. Since $1 \ge A^2 \ge 1-(2\mathrm{r})^{-2}$ we have moreover for macroscopic distances almost exactly $A^2=1$ (see the remarks near II.12 and II.13) .
The term $(1+E^2)r_0$ instead of just r_0 in $b^2 = (1+E^2)r_0/r$ not necessarily means, that the gravitational force is $(1+E^2)$ times higher as in the common theory. If we have no other space regions, with a different factor to compare, we have to gauge now also our physical constants with, $(1+E^2)r_0$ which is, that we have to multiply r_0 in the standard theory with the factor $(1+E^2)$ and hence will get just $r_0 = mass$ instead of $r_0 = 2\,mass$. But if we have two regions in space-time with distinct values of E^2 , we will get different forces, i.e gravitation constants. One may get a such one, if setting $\beta = \Theta(t)t - h(r)$ (where Θ is the Heaviside function) , but then we will get a discontinuous time $\tau \;=\; t + \Theta(t)(E^2 + v(r))$. Below, we will provide a solution with a continuous time τ . But first we introduce a simple particle picture. The equation $d\,\beta = E\,dt - P\,dr$ is the Hamilton-Jakobi equation for the action $S = -\beta$ [Gs],

$$\frac{\partial S}{\partial t} = -E\,,\;\; \frac{\partial S}{\partial r} = P\,,\;\; \frac{dS}{dt} = -E + P\,\dot{r} = \mathscr{L} \quad , \tag{III.2.10}$$

with Lagrange function $\mathscr{L}$. So also ω is the energy density and m the momentum density but in the original coordinate system (see III.2.2). Originally we introduced β as the phase of the Kaluza-Klein dimension (II.7) . In complex notation the Kaluza-Klein dimension may be written as $z = \mathrm{e}^{iS} \equiv \mathrm{e}^{-i\beta}$, and from this we have

$$i\frac{\partial z}{\partial t} = E\,z\,,\;\; -i\frac{\partial z}{\partial r} = p\,z \quad . \tag{III.2.11}$$

So any related characteristic equations may be interpreted as the equations of motions of some “strange” particle.
Now continuing to look for a more general solution of III.2.2, III.2.3. We start with the ansatz III.2.7, but for arbitrary E . Obviously we have $0 \le \omega^2 < 1$, which leads to a Lorentz signature in the constant case above. The sign of E or ω is indeterminate and we choose it to be positive. We now get

$$\begin{gathered} \tau_{,r} = -a^{-2}\omega P \;\Rightarrow \\ \frac{\partial}{\partial r}\frac{\omega}{\sqrt{1-\omega^2}} = -P_{,t}\,,\;\; \frac{\partial}{\partial r}\frac{1}{\sqrt{1-\omega^2}} = -a^{-2}(\omega P)_{,t} \quad . \end{gathered} \tag{III.2.12}$$

Since $1 = 1/(1-\omega^2) - \omega^2/(1-\omega^2)$, a linear combination of the last two equations leads to

$a^2\omega P_{,t} = (\omega P)_{,t} \;\Rightarrow -x\,P_{,t}\omega = \omega_{,t}P$ with the general solution

$$\omega = \omega_a(r)|P|^{-x}\,,\;\; \Rightarrow\;\; P = \pm\left(\frac{\omega_a(r)}{\omega}\right)^{r/r_0} \;\text{for}\, r_0 > 0, \omega > 0 \quad . \tag{III.2.13}$$

with so some arbitrary function $\omega_a(r)$. If $r_0 = 0$ this means, that ω must be time independent and so also $P_{,t}$ and hence $P = u(r)t + v(r)$. But the last equation in III.2.12 then leads to $u=0$ and therefore $P_{,t} = 0 \Rightarrow \omega = \mathrm{const}$. So we have as a first result, that

for $r_0=0$ **only a solution with constant** ω **exists.** (III.2.14)

In general we now get with III.2.12and III.2.13 the first order PDE,

$$E_{,r}=-P_{,t}=-\frac{\partial P}{\partial E}E_{,t},\ \ \frac{\partial P}{\partial E}=\frac{\partial P}{\partial \omega}\frac{\partial \omega}{\partial E}=\frac{-rP}{r_0\omega}\cdot\frac{1}{(1+E^2)^{3/2}}=\frac{-r\,P}{r_0 E(1+E^2)} \qquad \text{(III.2.15)}$$

which implies that E (and so ω) is a conserved quantity along the characteristic

$$P\,v:=P\frac{dr}{dt}\ =\ -\frac{r_h}{r}E\,,\ \ r_h:=\frac{r_0}{1-\omega^2}\equiv r_0(1+E^2) \qquad \text{(III.2.16)}$$

To determine the arbitrary function $\omega_a(r)$ in III.2.13 we have a look at the scale factors. They have the equivalent form as in the constant case above

$$ds^2=b^2dt^2-q^2dr^2$$, where

$$b^2=1-\frac{1}{r}\frac{r_0}{1-\omega^2},\ \ q^2=P^2(1-\omega^2/a^2)+f^2=P^2\frac{b^2}{a^2}(1-\omega^2)+f^2 \qquad \text{(III.2.17)}$$

Assume that we have a solution with $\lim_{t\to\infty}\omega=\omega_\infty>0,\ \ \omega_\infty=\text{const.}$ then

$$b_\infty^2=1-\frac{1}{r}\frac{r_0}{1-\omega_\infty^2},\ \ q_\infty^2=P^2\frac{b_\infty^2}{a^2}(1-\omega_\infty^2)+f^2=\left(\frac{\omega_a(r)}{\omega_\infty}\right)^{2r/r_0}\frac{b_\infty^2}{a^2}(1-\omega_\infty^2)+f^2\ \ .$$

To obtain the Schwarzschild metric in this limes, we set $q_\infty^2=b_\infty^{-2}$ and get

$$P^2=(b_\infty^{-2}-f^2)\left(\frac{\omega_\infty}{\omega}\right)^{2r/r_0}\frac{a^2}{b_\infty^2}(1+E_\infty^2),\ \ q^2=(b_\infty^{-2}-f^2)\left(\frac{\omega_\infty}{\omega}\right)^{2r/r_0}\frac{b^2}{b_\infty^2}\frac{(1+E_\infty^2)}{(1+E^2)}+f^2\ \cdot \qquad \text{(III.2.18)}$$

Now the characteristic equation is completely defined and could be principally integrated

$$\int_{r_A}^{r}r'P(E,r')dr'\ =\ -r_hEt\ \ . \qquad \text{(III.2.19)}$$

Solving this equation for r_A we get the solution of the PDE with initial values $\omega_0(r)$ due to solving $\omega=\omega_0(r_A)$. We now have to analyze, for which values or the time parameter a unique solution exists and that we can recover $\lim_{t\to\infty}\omega=\omega_\infty>0$ .We further expect and demand $E_{,t}\leq 0$ and since $E_{,t}=-v\,E_{,r}=\frac{r_h}{r}\frac{E\,E_{,r}}{P}$ this is

$$sign(E^2)_{,r}=-sign\,P\ \ , \qquad \text{(III.2.20)}$$

and because we defined $E\geq 0$ it's equivalent to $sign\,E_{,r}=-sign\,P$. For outgoing solutions we have $v\geq 0$ and hence $P\leq 0$ and $E_{,r}\geq 0$ and for inward directed the

opposite signs. This relations specially have to hold for the initial values at $t=0$, which we have so to choose in this way.
For ω_∞ we select one of the two values $0, 1/2$, which we received in the stationary case and since we want $\omega_\infty>0$ we only have $\omega_\infty^2=1/2$. So the parameters are.

$$\begin{aligned}
&b_\infty^2=1-\frac{2\mathrm{r}_0}{r}, \quad P=\pm\sqrt{2}\left(\frac{1}{2\omega^2}\right)^{r/2\mathrm{r}_0}\frac{a}{b_\infty^2}A,\\
&q^2=\left(\frac{1}{2\omega^2}\right)^{r/r_0}\frac{b^2}{b_\infty^4}\frac{2\mathrm{A}^2}{(1+E^2)}+f^2, \text{ with } \quad A^2:=1-b_\infty^2 f^2
\end{aligned} \tag{III.2.21}$$

As above P keeps real iff III.2.9 holds. Now we put $\omega=\omega_0(r_A)$ into III.2.19 , which implicit defines some function $r_A=g_A(t,r)$, which we simply just also denote with $r_A(t,r):=g_A(t,r)$, and so an equation of type $G(r, r_A(t,r))=0$. We build the partial derivatives of this equation with respect to t and r and get

$$\begin{aligned}
r_{A,r}\cdot\left(-P(E,r_A)+G(r,r_A,t)\right) &= -Pr, \quad r_{A,t} = -v\, r_{A,r}\\
G(r,r_A,t) := E'_0(r_A)&\left(\int_{r_A}^{r} r'\frac{\partial P(E,r')}{\partial E}dr'+t\frac{d(r_h E)}{dE}\right) = \\
= E'_0(r_A)&\left(\frac{-\int_{r_A}^{r}(r')^2 P\,dr'}{r_0 E(1+E^2)}+r_0(1+3E^2)\cdot t\right)
\end{aligned} \quad . \tag{III.2.22}$$

Now we receive $v>0,\ r_{A,t}<0,\ r_{A,r}>0$ for $r\geq r_A>2\mathrm{r}_0$, if we choose for P the negative sign and, applying III.2.20, $E_0\geq 0$. This proves the uniqueness of the solution (if there is one), because for any finite (t,r) exists (at most) exact one starting point r_A at $t=0$. For incoming waves $P>0, \omega_0'<0$ we can't draw this conclusion, because the middle term $\omega_0'\cdot dP/dE$ in the brackets above is always positive and have for $P>0$ the opposite sign of the two other one. So in the following we will restrict us to the case of outgoing waves and have (for $r\geq 2\mathrm{r}_{0,}$ $t>0$

$$\begin{aligned}
&E\geq 0,\ P=P=-\sqrt{2}\left(\frac{1}{2\omega^2}\right)^{r/2\mathrm{r}_0}\frac{a}{b_\infty^2}A\leq 0,\ \ E_{,t}\leq 0,\ E_{,r}\geq 0,\ \ P_{,t}\geq 0,\ P_{,r}\leq 0,\\
&\mathrm{v}\geq 0,\ \ \mathrm{v}_{,t}\leq 0,\ \mathrm{v}_{,r}\geq 0, \text{ since } d\mathrm{v}/d\omega>0
\end{aligned} \quad . \tag{III.2.23}$$

At $r=2\mathrm{r}_0$ the amount of the momentum P becomes infinite and hence $\mathrm{v}=0$. The characteristic starting there is the curve $r=2\mathrm{r}_{0,}$ parallel to the time axis and hence $\omega=\omega_0(2\mathrm{r}_0)=\omega_\infty=1/2$ along it. Now we have to show, that we receive any point in the area $t>0, r>2\mathrm{r}_0$. For microscopic masses of size $2r_0\ll 1$, we get the problem of a vanishing A^2 near $r=1$ (see below III.2.8) . We will shortly discus this at the end. To sketch a proof for the existence of a solution for all $t>0$, we now restrict us to macroscopic masses and distances $r\geq 2\mathrm{r}_0\gg 1$ and so $A^2>\text{const}>0$ holds. Moreover we can set with high accuracy $A^2=1$ and the terms in III.2.18 become

$$P^2=2b_\infty^{-4}a^2(2\omega^2)^{-r/r_0},\ \ q^2=2b_\infty^{-4}(2\omega^2)^{-r/r_0}(a^2-\omega^2)+f^2,\ \ b_\infty^2=1-2\mathrm{r}_0/r \tag{III.2.24}$$

and the characteristic equation therefore

$$\begin{aligned} r_h E t \;&=\; I(r, r_A) := \sqrt{2}\int_{r_A}^{r} r'\exp(-\lambda r'/2\mathrm{r}_0)\frac{\sqrt{1-r_0/r'}}{1-2\mathrm{r}_0/r'}dr' \\ &=\; \eta\cdot(2\mathrm{r}_0)^2\cdot J(r, r_A) := \eta\cdot\int_{r_A}^{r} r'\frac{\exp(-\lambda r'/2\mathrm{r}_0)}{1-2\mathrm{r}_0/r'}dr' \\ \lambda &:= \log(2\omega^2),\;\; 1\le\eta\le\sqrt{2},\;\; r\ge r_A > 2\mathrm{r}_0 \end{aligned} \quad . \qquad \text{(III.2.25)}$$

The integrand III.2.25 is singular of type $1/r$ at $b_\infty = 0 \Rightarrow r = 2\mathrm{r}_0$ and hence the integral diverges there. On the other side for large r_A (and $r > r_A$) the integrand get the form $r\exp(-\lambda r)$ and $I(\infty, r_A) \approx (\lambda r_A + 1)\exp(-\lambda r_A)/\lambda^2$ exists. This means, for $r\to\infty$ all characteristics become straight lines parallel to the r-axis i.e. they ending at some finite time t . If $I(\infty, r_A) = r_h E t$ have a bounded solution $r_A(t) < \infty$, than we conclude from $r_{A,t} < 0,\; r_{A,r} > 0$ (III.2.22), that

$$\lim_{t\to\infty}\lim_{r\to\infty} r_A(t, r) = \lim_{r\to\infty}\lim_{t\to\infty} r_A(t, r) = 2\mathrm{r}_0 \quad . \qquad \text{(III.2.26)}$$

To be a bit more precise: $J(r, r_A)$ could be explicitly calculated as

$$\begin{aligned} J(r, r_A) &= \int_{Y_A}^{Y(r)} (y+2+y^{-1})\mathrm{e}^{-\lambda(y+1)}dy,\;\; Y(r) = \frac{r}{2\mathrm{r}_0} - 1,\;\; Y_A = Y(r_A) \\ &= e^{-\lambda}\left(\mathrm{Ei}(-\lambda y) - \lambda^{-2}((y+2)\lambda+1)\exp(-\lambda y)\right)\Big|_{Y_A}^{Y(r)} \;\Rightarrow \\ J(\infty, r_A) &= -e^{-\lambda}\left(\mathrm{Ei}(-\lambda Y_A) - \lambda^{-2}((Y_A+2)\lambda+1)\exp(-\lambda Y_A)\right) \end{aligned} \quad . \qquad \text{(III.2.27)}$$

Now $J(\infty, r_A) \ge 0$ (because $\mathrm{Ei}(-y) < 0$ for all $y > 0$) and λ depends from the initial values and so from r_A . Since we defined $\omega_\infty = \omega_0(r_A = 2\mathrm{r}_0) = 1/2$, we conclude monotone increasing initial values with range $\omega_\infty^2 = 1/2 \le \omega_0^2 \le 1$ and therefore $0 \le \lambda = \log(2\omega^2) \le \sqrt{2}$ and λ increases monotone with r_A . We rewrite the last equation $I(\infty, r_A) = \eta(2\mathrm{r}_0)^2 J(\infty, r_A) = r_h E t = 2\mathrm{r}_0 E(1+E^2)t$ as

$$\frac{t}{2\mathrm{r}_0} = \eta\frac{J(\infty, r_A)}{E(1+E^2)} \quad . \qquad \text{(III.2.28)}$$

Since from monotony we have $r_A\to\infty \;\Rightarrow Y_A\to\infty$ for $\omega\to 1 \Leftrightarrow \lambda\to\sqrt{2} \Leftrightarrow E^2\to\infty$ the factor $J(\infty, r_A)$ at least keeps finite for $\omega\to 1$ and hence $t = 0$ is the only solution for $\omega\to 1$. This implies , that for $t > 0$ we always have $\omega < 1$ and therefore r_A, Y_A and E are bounded . For $\omega^2\to 1/2 \Leftrightarrow \lambda\to 0$ also Y_A tends to its minimum $Y_A = 0,\; r_A = 2r_0$, therefore $J(\infty, r_A) \to 1/\lambda^2$ and III.2.28 becomes

$$\frac{t}{2\mathrm{r}_0} = \frac{\eta}{2}J(\infty, r_A\to 2\mathrm{r}_0) \to \frac{\eta}{\lambda^2} \to \infty$$

and proves that indeed also $\lim_{t\to\infty}\lim_{r\to\infty} r_A(t, r) = 2\mathrm{r}_0$ holds. The singularity at $r = 2\mathrm{r}_0$ guarantees, that we receive a solution everywhere in $[t\ge 0]\cap[r\ge 2\mathrm{r}_0]$.

Now we present a, in some way “natural”, candidate for the initial values of the desired kind, i.e. strictly monotone increasing with $\omega_\infty^2 = 1/2 = \omega_0^2(r = 2r_0) \le \omega_0^2 \le 1$.

$$\omega_0^2 = a^2 = 1 - r_0/r\,, \;\Rightarrow\; E_0^2 = r/r_0 - 1 \quad . \tag{III.2.29}$$

With these initial values, we have at $t=0$ the pure three-dimensional metric

$$b^2 = 0,\; q^2 = f^2 \Rightarrow\; ds^2 = -f^2 dr^2 - r^2 d\Omega \quad , \tag{III.2.30}$$

where even the radial dimension is iny of size r^{-2} and so we have almost the metric of a two-dimensional sphere $ds^2 = -r^2 d\Omega$. For large r the proper length for III.2.30 becomes $dl = r^{-1} dr \Rightarrow l = \log(r),\; r = \exp(l)$ and so even extreme large radial distances become very small in terms of it. III.2.29 can be derived from a flat Minkowski metric

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega$$

where the time coordinate is bound to the radial through $dt^2 = (1 - f^2) dr^2$. This relation is, if setting $t = \beta$, identical, to the one we found in II and becomes II.16 for $r_0 = 0$.

For $t \to \infty$ we get from construction $b^2 \to b_\infty^2 = 1 - 2r_0/r\,,\; q^2 \to b_\infty^{-2}$, i.e the Schwarzschild metric. Since $1 > \lim_{r\to\infty} \omega^2 > 1/2$ for any time $t > 0$ we have also $\lim_{r\to\infty} b^2 \to 1$. Further we conclude for small r_0/r from III.2.24

$$\begin{gathered} P^2 \approx 2(2\omega^2)^{-r/r_0},\; q^2 \approx 2(2\omega^2)^{-r/r_0}(1 - \omega^2) + f^2, \\ b_\infty^2 = 1 - \frac{2r_0}{r},\; b^2 = 1 - \frac{(1+E^2) r_0}{r} = 1 - \frac{r_A}{r} \end{gathered} \tag{III.2.31}$$

Since q^2 decreases exponentially as $r \to \infty$, we receive $q^2 \to f^2 \to 0,\; b^2 \to 1$. The same holds also on any characteristic curve C_A through $r_A > 2r_0$. Generally $b^2 \to 1$ holds on any curve C , with $\dot t := dt/ds \ge 0,\; \dot r := dr/ds > 0$ but not $q^2 \to 1$. The curves C_q along which $dq^2/dt = 0$ holds, divide the local space into two regions. On a curve C , as above, one reaches for arc-length $s \to \infty$ the usual flat space , $b^2 \to 1, q^2 \to 1$, if the velocity v_C on the curve is smaller than the velocity v_q on C_q , If $\mathrm{v}_C > \mathrm{v}_q$ the curve leads to the sphere $b^2 \to 1, q^2 \to 0$. For v_q we have the equation

$$\begin{gathered} d\,q^2(\omega, r) = q^2_{,\omega}\, d\omega + q^2_{,r}\, dr = q^2_{,\omega}\, \omega_{,t} dt + (q^2_{,\omega}\, \omega_{,r} + q^2_{,r})\, dr \;\Rightarrow \\ \mathrm{v}_q = \frac{\mathrm{v}}{1+u},\; u = \frac{q^2_{,r}}{q^2_{,\omega}\,\omega_{,r}} \end{gathered} .$$

Since $\omega_{,r} \ge 0$ and, using III.2.21 or III.2.18 , $q^2(\omega, r)_{,\omega} \le 0,\; q^2(\omega, r)_{,r} \le 0$ we find $u \ge 0$ hence $0 \le \mathrm{v}_q \le \mathrm{v}$ follows. For small r_0/r we further get $q^2_{,r}/q^2_{,\omega} \sim r_0/r$ and $r\omega_{,r} = r(r_A/r^2 - r_{A,r}/r) \sim r_A/r$ and therefore $u \sim r \cdot r_0/r_A \Rightarrow \mathrm{v}_q/\mathrm{v} \to 0$ holds.
We combine the results for the metrics. For initial values III.2.29 we have the limits

$$\begin{aligned}
&t=0,\ r\geq 2\mathrm{r}_0 \quad ` &&: ds^2= \quad - f^2 dr^2 - r^2 d\Omega, \quad f^2=\frac{1}{r^2 a^2}\left(\frac{3}{2}\frac{r_0}{r}-1\right)^2 \\
&t\rightarrow\infty, r>2\mathrm{r}_0=\ \text{const}: && ds^2=\ b^2 dt^2 - b^{-2} dr^2 - r^2 d\Omega, \quad b^2=1-2\mathrm{r}_0/r \\
&r\rightarrow\infty, t>0=\ \text{const}: && ds^2=\ dt^2 - f^2 dr^2 - r^2 d\Omega, \qquad f^2\rightarrow r^{-2}\rightarrow 0 \\
&\text{Gneral on a curve C, for arc length } s\rightarrow\infty \text{ and with } \dot{t}>0,\ \dot{r}>0,\ 0\leq\eta^2\leq 1, \\
& &&: ds^2= \quad dt^2 - \eta^2 dr^2 - r^2 d\Omega
\end{aligned} \tag{III.2.32}$$

where $\eta^2=1$ if $\mathrm{v}_C<\mathrm{v}_q$ and $\eta^2=r^{-2}\rightarrow 0$ if $\mathrm{v}_C>\mathrm{v}_q$.
On the null-geodesics of the manifold (light-like curves), we have the velocity

$$\begin{aligned}
&\mathrm{v}_L^2=b^2 q^{-2}, \quad q^2=A^2 P^2 \frac{b^2}{a^2}(1-\omega^2)+f^2 \Rightarrow \\
&\left(\frac{\mathrm{v}}{\mathrm{v}_L}\right)^2=\left(\frac{r_h}{r}\right)^2 \frac{E^2 q^2}{P^2 b^2}=\left(\frac{r_0}{r}\right)^2 \frac{E^2}{a^2(1-\omega^2)}\left(1+\frac{f^2}{q^2-f^2}\right)
\end{aligned}$$

With the initial values III.2.29 we write $\omega^2 = 1-r_0/r_A$, $E^2 = r_A/r_0-1$ and hence

$$\left(\frac{\mathrm{v}}{\mathrm{v}_L}\right)^2=\left(\frac{r_0}{r}\right)^2\frac{(r_A/r_0-1)}{a^2 r_0/r_A}=\frac{r_A}{r}\frac{(r_A-r_0)}{(r-r_0)}\left(1+\frac{f^2}{(q^2-f^2)}\right) \quad .$$

At $r=r_A$ (i.e. $t=0$) we have $q^2=f^2$ and hence $|\mathrm{v}/\mathrm{v}_L|=\infty$. This is just the consequence from the fact that the characteristics start with non-zero velocity at $t=0$, but, due to $b^2(t=0)=0$, the speed of light v_L is zero. The characteristics become for $t\rightarrow\infty$ parallel to the t -axis. Since $\lim_{t\rightarrow\infty} q^2\rightarrow b_\infty^{-2}$ and f^2 is at any macroscopic distance negligible, we receive for small r_0/r at $t\rightarrow\infty$ the ratio

$$\left|\frac{\mathrm{v}}{\mathrm{v}_L}\right|\approx\frac{r_A}{r}=1-b^2=\frac{r_0}{r}(1+E^2)\approx\frac{2\mathrm{r}_0}{r}\ll 1 \quad . \tag{III.2.33}$$

For the initial values III.2.29 , we now give an approximate solution of III.2.27 and so of the characteristic equations**.** Assume $\lambda=\log(2\omega^2)$ is small. Since $2\omega^2=2-2\mathrm{r}_0/r_A$ we get $Y_A=\frac{r_A}{2\mathrm{r}_0}-1\approx\lambda$ and $E^2=\frac{r_A}{r_0}-1\approx 1+2\lambda$. Define $\lambda Y(r)=\lambda\left(\frac{r}{2\mathrm{r}_0}-1\right)=\epsilon$ and use a λ , that $\epsilon\leq O(1)$ is at least not large. even for small r_0/r . Now we calculate with $\exp(-\lambda Y_A)\approx\exp(-\lambda^2)\approx 1$

$$\begin{aligned}
J(r,r_A)&\approx(1-e^{-\epsilon}(1+\epsilon))/\lambda^2+2(1-e^{-\epsilon})/\lambda+\log(\epsilon)-\log(\lambda)+O(1)\\
&\approx(1-\exp(-\epsilon)(1+\epsilon))/\lambda^2=Y_r^2\frac{1-\exp(-\epsilon)(1+\epsilon)}{\epsilon^2}:=U(\epsilon)\cdot Y_r^2 \quad .
\end{aligned} \tag{III.2.34}$$

The factor $U(\epsilon)$ is monotone decreasing and has the range $1/2=U(0)\geq U\geq 0$. For $\epsilon\leq O(1)$ this factor U is of non-vanishing order ($U(10)\approx 10^{-2}$) . Therefore we receive with III.2.25 $\eta(2\mathrm{r}_0)^2 J(r,r_A)=2\mathrm{r}_0 E(1+E^2)t$ (as for III.2.28) and $r/r_0\gg 1$

$$\frac{t}{2r_0} = \eta \frac{J(r, r_A)}{E(1+E^2)} \approx \frac{\eta_\epsilon}{2(1+2\lambda)} \cdot \left(\frac{r}{2r_0}\right)^2 \approx \frac{\eta_\epsilon}{2} \cdot \left(\frac{r}{2r_0}\right)^2 \quad . \tag{III.2.35}$$

where for the factor $\eta_\epsilon = U\eta$ we have approximately $1 \geq 2\eta_\epsilon > 10^{-2}$ if $0 \leq \epsilon \leq 10$.
With $E^2 = 1+2\lambda$ we also have for energy the approximation

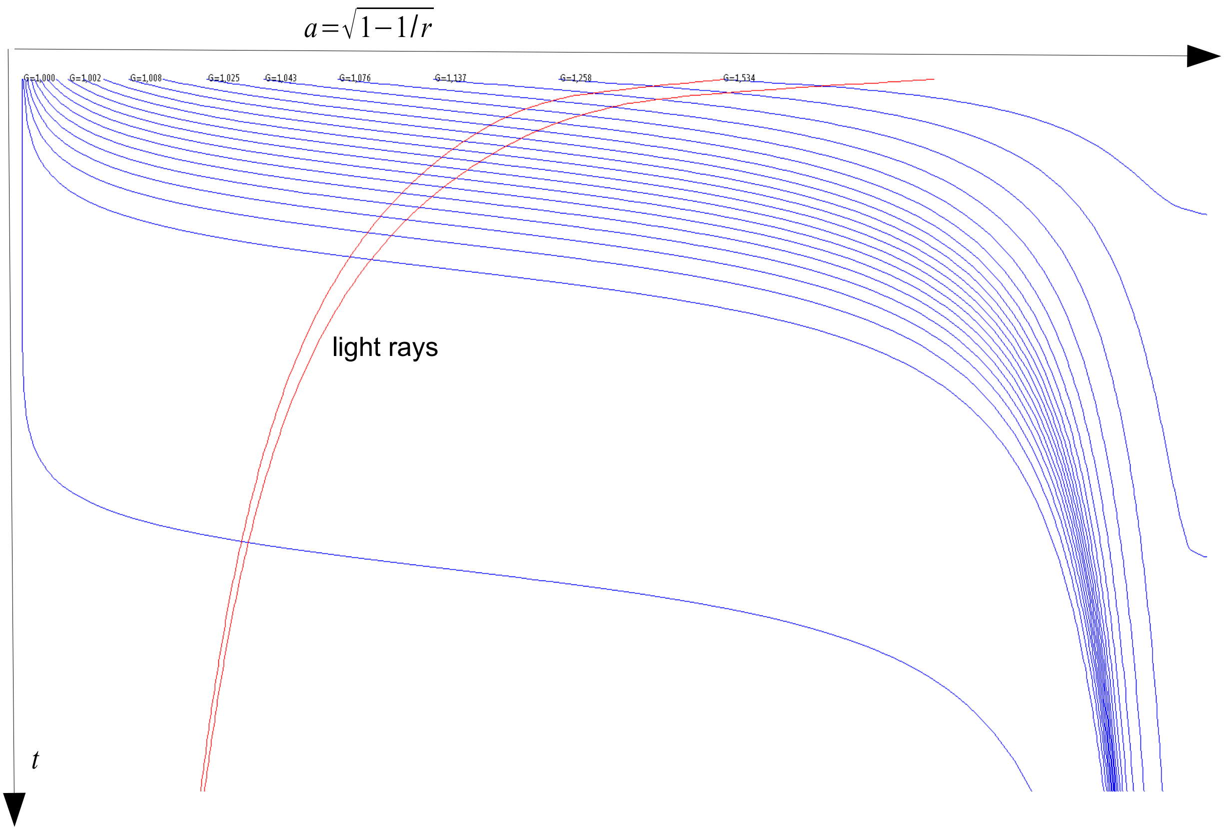
$$E^2 \approx \frac{\eta_\epsilon}{4} \frac{r^2}{t \cdot r_0} \tag{III.2.36}$$

III.3 Conclusions

We now offer some consequences arising from the obtained metric.

i,) Cosmic redshift

If we neglect $f^2 \approx 0$, we have from III.2.21 $q^2 = (2\omega^2)^{(-r/r_0)} b_\infty^{-4} b^2 / (1+E^2)$ and so for the velocity of a light ray (in this coordinate system) $v_L^2 = (2\omega^2)^{(r/r_0)} b_\infty^4 (1+E^2)$. From this we conclude $sign(\partial v_L^2 / \partial t) = sign(\partial \omega^2 / \partial t) < 0$ i.e. the speed of light decreases with time. So any emitted photon is slower than it's predecessor, which creates a red-shift, as from a departing source. So the provided metric, used as cosmological model, leads to an expanding universe. The following picture shows some numerically calculated characteristics and light rays. As radial parameter $a = \sqrt{(1 - r_0/r)}$ is used.



ii.) Newtonian forces at large distances

As we seen, the velocity of the characteristic is small at a large distance r and also the temporal change of ω . So it needs some time span until the geometry of time space becomes nearly flat Schwarzschild-like and there will be a difference in the Newtonian force, even at a relative large time. To estimate this force for small $2r_0/r$, we use III.2.31and assume a momentum $P^2 \approx 2(2\omega^2)^{-r/r_0} = 2/\sigma^2$, where $\sigma \geq 1$ is of size $\log(\sigma) = O(1)$ (e.g. $1 \leq \sigma \leq e^4$, note $\sigma = 1 \Rightarrow q^2 = P^2/2 + f^2 \approx 1$ indicates the flat space). Thus we have $2\omega^2 = \sigma^{2r_0/r} \approx 1 + \log(\sigma) 2r_0/r$ and obtain

$$. \quad 2(1 - \frac{r_0}{r_A}) = 2\omega^2 \approx 1 + \log(\eta)\frac{2r_0}{r} \Rightarrow \; b_\infty^2(r_A) = 1 - \frac{2r_0}{r_A} = \log(\eta)\frac{2r_0}{r} \ll 1$$

Hence the corresponding characteristic still starts near $r_A = 2r_0(1 + O(2r_0/r)) \approx 2r_0$. We further have on this characteristic $E^2 = r_A/r_0 - 1 \approx 1$ and so for the velocity

$$\mathrm{v} = \frac{-r_0}{r}\frac{2}{P} \approx \sqrt{2}\frac{r_0}{r}\sigma \qquad \text{(III.3.1)}$$

Let $D_A := -P(E, r_A)$. Because $b_\infty^4(r_A) \sim (r_0/r)^2$ we have $D_A \sim (r/r_0)^2 \gg 1$. For III.2.22 we first calculate (note $E_0{}'(r_A) = r_0^{-1}$)

$$G(r, r_A, t) \; = \; D\frac{r^3 - r_A^3}{2r_0^2} + 4t, \;\; D := -P(\hat{r}), \hat{r} \in [r_A, r] \; \Rightarrow D_A > D > \sqrt{2}/\sigma$$

and get as an estimation

$$. \quad -Pr = \sqrt{2}r/\sigma = r_{A,r} D_A(1 + 4\frac{t}{D_A} + \frac{D}{2r_0^2 D_A}(r^3 - r_A^3)) \; \Rightarrow r_{A,r} < (\frac{r_0}{r})^2$$

From this we conclude, that for $q^2 \approx P^2/2$ of size $1/\eta^2$ the variation of the energy is already very small $E_{,r}^2 = r_0^{-1} r_{A,r} \sim r_0/r^2 \Rightarrow E_{,t}^2 = -\mathrm{v} E_{,r}^2 \sim r_0^2/r^3$. Specially we find also $b_{,r}^2 = \frac{2r_0}{r^2}(1 + O(\frac{r_0}{r})), \; -b_{,t}^2 = \frac{2r_0}{r}E_{,t}^2 = O(r_0^3/r^4)$. Since for small velocities and a $b^2 \approx 1$ we have also $ds^2 \approx dt^2$ (up to $O(r_0/r)$) and we may use the Newtonian approximation for the static metric

$$\ddot{r} \approx -\Gamma_{tt}^r = \frac{1}{2} \cdot g^{rr} \cdot g_{tt,r} = -\frac{b_{,r}^2}{2q^2} \approx -\sigma^2\frac{r_0}{r^2} \, . \qquad \text{(III.3.2)}$$

This force is σ^2 times stronger than the usual Newton force, For $t \rightarrow \infty$, σ^2 must tend to one, and we my interpret σ^2 as a time dependent monotone decreasing gravitation constant.
To estimate the distance, between two coordinates with different values for $q^2 = \sigma^{-2}$, we look at III.2.34 $\frac{t}{2r_0} \approx \frac{\eta}{2}U(\epsilon) \cdot (\frac{r}{2r_0})^2$ and note $\epsilon = \log(\sigma)$, So we can estimate the mass r_0 (remember r_0 is here indeed the mass and not $r_0/2$ as in usual GR), which generates forces at a distance r , which are for $t \leq r^2$ larger than σ -times the Newton force as

$$r_0 \approx \frac{\eta}{4} \cdot U(\log(\sigma)) \approx U(\log(\sigma))/4 \approx 0.1 \quad (\text{for } \sigma \approx 1)$$.

If we set e.g. $r = 10^5 Ly,\ t = 10^{10} y$ (distance of a galactic periphery and a assumed galactic age), than $r_0 \approx 0.1$ is measured in light years and we get roughly the mass of $10^{11} - 10^{12}$ solar masses, i.e a galactic mass . At smaller times bigger forces acting. In the limit $r/r_0 \to \infty$ we even get with III.2.32 $\sigma^2 \to r^2$ and so the constant force

$$\ddot{x} = F_N \approx \approx -2r_0 \quad .$$

Setting this force equal to the centrifugal forces, leads to a rotation velocity at radius R of $v_{rot}^2 \sim R$, where for Kepler orbits ($\sigma = 1$) one has $v_{rot}^2 \sim R^{-1}$. The observed galactic orbital speeds for outer regions are of the kind $v_{rot}^2 \sim 1$ and so lay exactly between the both extreme values of the metric discussed here, So with this metric we get also effects like the one associated to dark matter.
But we may also interpret this limit in another way. For a very small mass r_0 , e.g. a atomic or baryonic mass, we already receive at it's boundary distance R (e.g. its charge radius) with typically $R/r_0 > 10^{35}$, at this "far limit". So the last equation $\ddot{x} = -2r_0$ defines some confinement like forces., which decay with time.

III.4 Final remarks

Since the obtained time dependent metric is not translation invariant, the main remaining question is, what kind of events triggers the start of the described process, that is, what does $t = 0$ mean. The first which comes into mind is the begin of the universe. One other approach would be to continue the characteristics back in time. This will lead to the same type of equations (and difficulties) as we will get for changing the sign of p and we not want to analyze here. But we are also able to extend the obtained solution to $t < 0$ by a reflection of the characteristics at $t = 0$, This accords to the mappings $t \to -t,\ E \to -E$ or $t \to -t,\ p \to -p$ or $E \to -E,\ p \to -p$, respectively. This mappings not influence the metric and we get $g(t, r) = g(-t, r)$. So at all finite spatial locations r , the metric becomes, at least in the far future and in the far past, the Schwarzschild metric. This ansatz now suggests the picture, that some mass or mass distribution first – in far past- acts on space-time as usual, but starts manipulating the geometry as described until $t = 0$. Then the inverse process leads again to the usual Schwarzschild metric. This again is some hint that the creation or annihilation time of any matter/particles may be the source of such effects.

IV The complex Space.

In this section we want to consider the embedding provided in section II in the context of Kähler geometry. The other sections are not dependent on the content of this one, so we think of this section as a proposal for people with a deeper knowledge in this subject to take a closer look at these kinds of spaces.
The 10 dimensional space carries the natural complex structure, which makes it a Kähler manifold [Hu,Mor]. The metric is the standard metric on $C^{1,4}$, a complex Lorentz (or Minkowski) metric

$$\langle u, v \rangle = u_0 \cdot \bar{v}_0 - \sum_1^4 u_i \cdot \bar{v}_i,$$
$$d^2 s = \langle dz, d\bar{z} \rangle$$.

The manifold

$$M = \{ z \in C^{1,4} : \langle z, z \rangle = -1 \}.$$

is a complex de-Sitter space. As the real one it is projective. Assuming the further restriction $|z_4| = 1$, the manifold is the product of a light cone and U(1). What we aim to do is an orthogonal projection of the Kaluza-Klein dimension onto the light cone (the Hopf fibration $S^1 \hookrightarrow S_2^9 \xrightarrow{p} CP^{1,3}$). Usually one derives a metric on a projective space in a coordinate system, which arises from a projection onto the first coordinate z_0 (the chart $U_0 := \{(z_0 : \ldots : z_n) \mid z_0 \neq 0\}$). For the embedding in the last section III, $|z_4|$ is always non-zero. Also, for this reason, but not only, using a projection on this coordinate, seems the better choice. This will give us the projective space $CP^{1,3}$ .The metric induced from $C^{1,4}$ is (see Appendix D)

$$ds^2 = \frac{1}{1-u^2} \cdot (\langle du, du \rangle + \frac{|\langle u, du \rangle|^2}{1-u^2}) \quad , \tag{IV.1}$$

where $u_j = \frac{z_j}{z_4}$, for $j = 0,1,2,3$, $u^2 := \langle u, u \rangle_M$ and $\langle .,. \rangle_M$ is the complex Lorentz metric.

Metric IV.1 corresponds to the real de-Sitter metric (see [Dr]) in projective coordinates, in the same way as the Fubiny-Study metric corresponds to the real projective metric of the ball. It looks almost like the usual metric on the hyperbolic space $\mathbb{C}H^4$ [Go], but with the Minkowski scalar product <,> "inside", instead of the Euclidean [3] . Denote the metric on the tangent bundle now as:

$$g(v, w) = \frac{1}{1-u^2} \cdot (\langle v, w \rangle + \frac{|\langle v, w \rangle|^2}{1-u^2}), \quad v, w \in T_u(M) \tag{IV.2}$$

Using Einstein sum conventions, IV.1 reads as

$$d^2 s = h_{\mu\nu} du^\mu d\bar{u}^\nu, \; h_{\mu\nu} = g(\partial / \partial u_\mu, \partial / \partial \bar{u}_\nu) \quad ,$$

$$h_{\mu\nu} := p(\eta_{\mu\nu} + p\bar{u}_\mu u_\nu) \; , \; \eta_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1), \; p = (1-u^2)^{-1} \quad .$$

The metric IV.1 / (IV.2) is Kähler, with Kähler potential ϕ (for definition see [Hu,Mor,Go])

$$\phi = \log(1-u^2) \,, \quad h_{\mu\nu} = \frac{d^2 \phi}{\partial z_\mu \partial \bar{z}_\nu} \quad . \tag{IV.3}$$

For calculating the determinant we write the hermitian matrix h as

$$h = p\eta(1 + p\sigma), \; \sigma^\mu_\nu = n^{\mu\lambda} \bar{u}_\lambda u_\nu = \bar{u}^\mu u_\nu, \; n^{\mu\lambda} n_{\lambda\nu} = \mathbf{1} \; (= diag(1,1,1,1)) \quad .$$

Therefore

$$\det h = \det(p\eta) \cdot \det(1 + p\sigma) = -p^4 \cdot \det(1 + p\sigma) \quad .$$

The second factor is calculated using

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3 Due to the definition in [Vr] it is $\mathbb{C}P^{1,3}$. . Note the opposite sign of the metrics here and in the mathematical literature.

$$\det A = \exp(\operatorname{tr}\log(A)) \quad \text{(tr := Trace of the matrix)}$$

With $\sigma^2 = u^2 \cdot \sigma \Rightarrow \sigma^j = u^{2(j-1)} \cdot \sigma$ we get

$$\log(1+p\sigma) = -\sum_{j=1}^{n} \frac{(-p\sigma)^j}{j} = \frac{-\sigma}{u^2} \sum_{j=1}^{n} \frac{(-pu^2)^j}{j} = \frac{\sigma}{u^2} \log(1+pu^2) \quad \text{for} \quad 0 < |u^2| < 1$$

and $\log(1+p\sigma) = p\sigma$ if $u^2 = 0$. Now $tr(\sigma) = u^2$ and so

$$\operatorname{tr}\log(1+p\sigma) = \log(1+pu^2), \text{ if } |u^2| < 1 \quad .$$

Finally we get

$$\det(h) = -p^4 \cdot (1+pu^2) = -(1-u^2)^{-5}$$

and so $$\log(\det(h)) = \pm i\pi - 5\log(1-u^2) \quad .$$

The Ricci Tensor for Kähler manifolds [Hu,Mo] is simply given by

$$Ric_{\mu\nu} = -\frac{\partial^2 \log(\det(h))}{\partial z_\mu \partial \bar{z}_\nu} \Rightarrow Ric_{\mu\nu} = 5\frac{\partial^2 \log(1-u^2)}{\partial z_\mu \partial \bar{z}_\nu} \tag{IV.4}$$

Comparing this with IV.3 we see, that the metric is Kähler-Einstein $Ric_{\mu\nu} = 5 h_{\mu\nu}$.
The curvature is positive as it is for the classical real de-Sitter space (and so the name "complex de-Sitter space" is in all senses appropriate).
For any embedding into M endowed with the metric II.9 and with $|z_4| := \rho = \text{const.}$
$\Rightarrow u^2 = 1 - \rho^{-2}$ (like we have done in section III) the metric simplifies to

$$ds^2 = \rho^2 \cdot \langle du, du \rangle + |(\rho^2 \cdot \langle u, du \rangle)|^2 \tag{IV.5}$$

and if $\rho = 1$ or after a rescaling

$$ds^2 = \langle du, du \rangle + |\langle u, du \rangle^2| \tag{IV.6}$$

Because $u^2 = const \Rightarrow \langle u, du \rangle = -\langle du, u \rangle$ ($\rightarrow$ $\langle du, u \rangle$ is pure imaginary)
one may write also

$$ds^2 = \langle du, du \rangle - \langle u, du \rangle^2 \, .$$

A consequence of this condition is that the second summand in the metric does not contributes to the Kähler form and $det(h) = -1$.
For writing down the Kähler form we use upper-lower indices with respect to the Lorentz metric $u^\mu = \eta^{\mu\nu} u_\nu, \ldots$ and write the metric again as

$$ds^2 = h_{\mu\bar{\nu}} du^\mu d\bar{u}^\nu, \;\; h_{\mu\bar{\nu}} = (\eta_{\mu\nu} + u_\mu \bar{u}_\nu) \quad .$$

The Kähler form is now (up to some factor)

$$\omega = h_{\mu\nu} du^\mu \wedge du^\nu = du_\mu \wedge d\bar{u}^\mu + u_\nu \bar{u}_\mu du^\mu \wedge d\bar{u}^\nu = du_\mu \wedge d\bar{u}^\mu - (u_\mu d\bar{u}^\mu) \wedge (u_\nu d\bar{u}^\nu) \quad .$$

Obviously, the second summand vanishes. So the Kähler form is just the same, as for the flat metric (the complex Lorentz metric).

$$\omega = du_\mu \wedge d\bar{u}^\mu = dv_\mu \wedge dw^\mu, \quad u = v + iw,\ v, w \in \mathbb{R}^{1,3}$$

An embedding is called totally real (respectively Lagrangian in symplectic geometry), if the induced Kähler form vanishes everywhere [4] (Go, Oh). So any totally real embedding into the complex Minkowski space $C^{1,3}$, for which u^2 is constant, is also totally real as embedding into **M** equipped with the metric II.9.
The embedding provided in the previous section is not Lagrangian. Its induced Kähler form doesn't vanish. For constructing a Lagrangian embedding set

$$\begin{aligned} &u_0 = a e^{i\omega} \Rightarrow v_0 = a\cos(\omega), w_0 = a\sin(\omega), \quad a, \omega \in \mathbb{R} \\ &u_j = b_j e^{i\eta} \Rightarrow v_j = b_j \cos(\eta), w_j = b_j \sin(\eta), \quad b_j, \eta \in \mathbb{R}, j = 1,2,3 \\ &b^2 := \sum b_j b_j, \quad a^2 = b^2 + k^2, \quad k^2 := u^2 = constant, \quad b := \sqrt{(b^2)} \end{aligned} \tag{IV.7}$$

and therefore

$$dv_0 \wedge dw_0 = a\,da \wedge d\omega, \quad \sum dv_j \wedge dw_j = \sum b_j\,db_j \wedge d\eta = 1/2 \cdot \sum db_j^2 \wedge d\eta = b\,db \wedge d\eta \quad .$$

Because $b\,db = 1/2\,d(b^2) = 1/2\,d(a^2) = a\,da$ the Kähler form is $\omega = a\,da \wedge d(\omega - \eta)$ and so the Kähler form vanishes, if $\omega - \eta = f(a)$, which is what we demand in the following. For calculating the metric II.9 under this restriction, rename $b = r$ (for getting the "familiar" 2-D surface term $r^2 d\Omega$ in the metric) and receive

$$\begin{aligned} &du_\mu d\bar{u}^\mu = dv_\mu dv^\mu + dw_\mu dw^\mu = \\ &= a^2 d\omega^2 - r^2 d\eta^2 - r^2 d\Omega = k^2 d\omega^2 + r^2(d\omega^2 - d\eta^2) - r^2 d\Omega \quad . \end{aligned}$$

With $u_0 d\bar{u}_0 = a(da + ia\,d\omega),\ \sum_{j=1}^{3} u_j d\bar{u}_j = \sum_{j=1}^{3} r_j(dr_j + ir_j d\eta) = r(dr + ird\eta)$, II.9 finally becomes (use $ada = rdr$ for constant u^2)

$$ds^2 = a^2 d\omega^2 - r^2 d\eta^2 - r^2 d\Omega + (a^2 d\omega - r^2 d\eta)^2 \quad . \tag{IV.8}$$

The metric now diagonalizes with the substitution

$$d\psi = \frac{a^2+1}{a^2+1-r^2}(d\omega - d\eta) + d\eta$$

This DGL could always be integrated, due to $\omega - \eta = f(r)$. From the substitution one also immediately sees that $\psi - \eta$ is also just a function of ***r*,** and vice versa, r is just a function of $\psi - \eta$. Summarized, diagonalizing IV.8 yields

4 For a totally real embedding the induced metric has no imaginary part. On the other hand there are complex submanifolds (closed complex subspaces). If, for an embedding, the tangential space at a point is mapped into a complex subspace , this point is called a complex point. The Kähler angle (or angle of holomorphy,[Sc], [Go]) between two vectors measures this difference. It is always $\pi/2$ for totally real embeddings and zero at complex points.

$$ds^2 = a^2 \frac{1+a^2-r^2}{1+a^2}((1+a^2-r^2)\,d\,\psi^2 - (r/a)^2\,d\,\eta^2) - r^2\,d\,\Omega \qquad \text{(IV.9)}$$

Now if $a=r \Leftrightarrow k=0$, (as in section III; the condition $k^2 := u^2 = 0$ means, that we consider the complex light cone) equation IV.9 reduces to

$$ds^2 = \frac{r^2}{(1+r^2)}(d\,\psi^2 - d\,\eta^2) - r^2\,d\,\Omega \quad . \qquad \text{(IV.10)}$$

To calculate the geodesics of this metric, we use the Euler-Lagrange formalism with $\dot{x} := dx/dt$, and for an affine parameter t proportional to the arc length

$$S[x,\dot{x}] = \int \sqrt{(L(x,\dot{x}))}\,dt = min \Leftrightarrow S[x,\dot{x}] = \int L(x,\dot{x})\,dt = min \quad \text{holds [St].}$$

So the Lagrange function corresponding to IV.10 is

$$L = \frac{r^2}{(1+r^2)}(\dot{\psi}^2 - \dot{\eta}^2) - r^2 \cdot S\,, \quad \text{with the usual surface term } S = \dot{\theta}^2 + \sin^2(\theta)\,\dot{\phi}^2 \quad .$$

Writing $r = \sinh(\varsigma)$, where ς is some function of $\psi - \eta$, results in $r' := \frac{dr}{d\psi} = -\frac{dr}{d\eta}$, and the Euler-Lagrange equations for the variables ψ and η are

$$\frac{d}{dt}(\cosh^2(\varsigma)\cdot\dot{\psi}) = \cosh(\varsigma)\sinh(\varsigma)\cdot\varsigma'(\dot{\psi}^2 - \dot{\eta}^2 - S)$$
$$\frac{d}{dt}(\cosh^2(\varsigma)\cdot\dot{\eta}) = \cosh(\varsigma)\sinh(\varsigma)\cdot\varsigma'(\dot{\psi}^2 - \dot{\eta}^2 - S)$$

and so it follows

$$\cosh^2(\varsigma)\cdot(\dot{\psi} - \dot{\eta}) = const \quad .$$

Because ς is only a function of $\psi - \eta$, the last equation is solvable iff $\psi - \eta$ is constant. So on geodesics we have $r = constant$. Now putting this back into the metric IV.10 only the surface term remains. The corresponding geodesics are the great circles on the ball of radius r . Because $\psi - \eta$ is constant also $\omega - \eta$ is constant and this geodesics may be written as (using the equator as a great circle)

$$u_0 = r\,\mathrm{e}^{i(\omega(t)-\omega_0)},\ u_1 = r\cos(\varphi(t))\,\mathrm{e}^{i\omega(t)},\ u_2 = r\sin(\varphi(t))\,\mathrm{e}^{i\omega(t)},\ u_3 = 0$$

or $$u_0 = r\,\mathrm{e}^{i(\omega(t)-\omega_0)},\ u_1 = u_0\cos(\varphi(t)),\ u_2 = u_0\sin(\varphi(t)),\ u_3 = 0 \quad .$$

The phase difference ω_0 is the value of the function $f(r) = \omega - \eta$ at constant r. On the geodesic great circles any point is oscillating "wave-like" in higher dimensions. The curves are space-like and nothing like an "eigentime interval" or a "proper length" exists. The metric degenerates to the one of a 2-D surface, as it is for the light cone in a flat Minkowski space ($t^2 = r^2 \Rightarrow ds^2 = -r^2\,d\,\Omega$). There is no dependency on the geodesics between the point on the surface (the angle φ on the great circle) and the frequency ω . So for example one may set $\varphi = constant$ to get "oscillating" points or $\phi = \omega$ for stationary waves.

Summarized we have, that the geodesics of an Lagrangian embedding into the complex light cone ($u^2=0$) with coherent phases of the space-like dimensions, are arbitrary "waves" on the Minkowski light cone.
Now we calculate the volume form of the Lagrangian subspace (again for any *k*) . We have

$$u_j=x_j e^{i\eta} \Rightarrow e^{-2i\eta} du_j \wedge du_k = dx_j \wedge dx_k + i\left(x_j dx_i + x_i dx_j\right) \text{ and therefore}$$
$$e^{-3i\eta} du_1 \wedge du_2 \wedge du_3 = dV + idS \wedge \eta$$

where dV is the 3D volume form and dS the surface form,
$dV := dx_1 \wedge dx_2 \wedge dx_3, \; dS = x_3 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_1 + x_1 dx_2 \wedge dx_3 \Rightarrow dr \wedge dS = rdV$.

So, with $du_0=e^{i\omega}(da+i\,a\,d\omega),\; a^2=r^2+k^2,\; \omega-\eta=f(r)$ the volume form is

$$\begin{aligned} dVol &:= du_0 \wedge du_1 \wedge du_2 \wedge du_3 \\ &= e^{i(\omega+3\eta)}\left(ia\cdot d\omega \wedge dV + i\,da \wedge dS \wedge d\eta - a\cdot d\omega \wedge dS \wedge d\eta\right) \\ &= e^{i(\omega+3\eta)}\left(i\left(a\,d\omega - \frac{r^2}{a} d\eta\right) \wedge dV + a\,d\eta \wedge d\omega \wedge dS\right) \\ &= a\,e^{i(\omega+3\eta)}\left(rf' + i\left(\frac{k}{a}\right)^2\right) d\eta \wedge dV. \end{aligned}$$

We see that the volume form is identical zero if $\omega-\eta = f(r)$ = constant and k = 0. In this case the metric IV.9 degenerates again just to a 2-D surface, the space is a light cone. From the volume form we read of the Lagrange angle (the phase of the form) [Vr,An] as $\omega+3\eta+\operatorname{atan}\left(k^2/(rf'a)\right)$. A Lagrangian submanifold is minimal, iff this angle is constant [An]. But then from $\omega-\eta = f(r)$ it immediately follows, that ω and η could only depend on ***r*** and so the volume form vanishes also. In this case, an appropriate choice of the arbitrary function $f(r)$ would transform IV.9 to a 3D flat space metric. So the minimal Lagrangian submanifolds with $u^2=const$, are the light cone (with all metric structure in 2-D spheres) or the pure 3-D spatial space (the absence of "time-intervals" could be interpreted as anything "travels with infinite speed").

In complex coordinates, the parametrization II.7 looks like

$$\begin{aligned} &u_0=r\,e^{i\omega},\; u_j=x_j e^{i\eta},\, j=1,2,3,\; u_4=e^{i\beta(r)} \\ &\omega=\kappa(r)\sinh(t),\; \eta=\kappa(r)\cosh(t) \end{aligned} , \qquad \text{(IV.11)}$$

which corresponds in the projective space (omitting the 2-D spherical part) to

$$u_0=r\,e^{i\omega},\; \omega=\kappa(r)\sinh(t)-\beta(r) \quad u_r=r\,e^{i\eta},\; \eta=\kappa(r)\cosh(t)-\beta(r) \; .$$

Here the difference of the angles is $\omega-\eta=-\kappa\,e^{-t}$, which vanishes for $t\to\infty$. So in this limit, the embedding becomes a minimal Lagrangian embedding, the light cone. Anything is concentrated on 2-D spheres, the boundary of the space. Remember, this result holds true for any stationary, isotropic metric, for which II.10 is true. That "anything is concentrated" on a 2-D surface reminds us that the entropy of a black hole depends only on the surface area.

V Geodesics from Classical Kaluza-Klein Theory

The classical K-K theory

Up to now the Kaluza-Klein dimensions have been only used to result in the correct metrics, but not in their original sense to combine gravitation theory with electromagnetism. First have a look at the standard interpretation. For this we rename the five coordinates as usual to $x^\mu,\ \mu=0\ldots3,\ y:=x^4$ (sometimes the index 5 is used for the extra dimension) and use again the standard notation, i.e summation over repeated upper, lower indices from 0 to 3 and $a_\mu:=g_{\mu\nu}a^\nu$, where $g_{\mu\nu}$ are the components of a 4-D metric tensor.
The Kaluza-Klein line element is [WL, Le, Str, Bl, Du]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu - \Phi^2\cdot(dy+A_\mu dx^\mu)^2 \tag{V.1}$$

and the corresponding Lagrange function

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \Phi^2(\dot{y}+A_\mu\dot{x}^\mu)^{2,} \text{ where } \dot{x}:=dx/ds \tag{V.2}$$

Now make the essential assumption of the theory, that the fields (A_μ and Φ) do not depend on y (called the "cyclic" assumption). So the fifth coordinate y is cyclic and due to Noether's first theorem, there is a conservation law. The geodesic equation for this coordinate just becomes

$$\Phi(\dot{y} + A_\mu\dot{x}^\mu) = q \ \Rightarrow \Phi(dy + A_\mu dx^\mu) = q\,ds \tag{V.3}$$

where *q* is an integration constant (identified later as the ratio unit-charge per mass). Setting $\Phi=1$, calculating the other geodesics, and using II.9 leads to the familiar equation of a particle in an electromagnetic field [Bl,St].

$$\begin{aligned}&\ddot{x}^\sigma+\Gamma^\sigma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=q\cdot F^\sigma_\mu\dot{x}^\mu \quad \text{or with } u^\mu:=\dot{x}^\mu\\ &Du^\sigma:=\frac{du^\sigma}{ds}+\Gamma^\sigma_{\mu\nu}u^\mu u^\nu=q\cdot F^\sigma_\mu u^\mu\end{aligned} \tag{V.4}$$

A main problem of the Kaluza-Klein (or KK) theory is that the metric is in fact not really a metric, as the quantities A_μ are not coordinates of a four-vector. As the components $A_\mu=g_{4\mu}/g_{44}$ are functions of the metric components, their behavior under coordinate transformation differs essentially from that of coordinates of a four-vector. If the A_μ are defined to be vector-components, the metric "is not invariant under arbitrary 5-D coordinate transformations" [Bl], which means, it is not a metric! To bring this difference in line, one limits the allowed coordinate transformations. One demands that they must have the specific form $x^\mu\rightarrow\tilde{x}^\mu(x^\mu),\quad y\rightarrow\tilde{y}=y+h(x^\mu)$, for which g_{44} remains unchanged and so

$$A_\mu dx^\mu=g_{4\mu}dx^\mu/g_{44}\rightarrow\tilde{g}_{4\mu}d\,\tilde{x}^\mu/g_{44}+(\partial h/\partial x^\mu)d\,\tilde{x}^\mu=\tilde{A}_\mu d\,\tilde{x}^\mu+d\,\tilde{y}-dy \quad .$$

So under this kind of transformation, the "vector" $A_\mu dx^\mu$ "remains the same" in different coordinate systems modulo some gauge transformation, which does not affect electromagnetic theory. But, applying mathematical theory and methods on not well-defined "hybrids" like this one, is in some way suspect and must be done very carefully.

Equation II.9 is usually interpreted in correspondence to gauge invariance. If applying a gauge transformation, the trajectory through the 5'th dimension has to be changed to

keep the expression constant. The overall change may be interpreted as a coordinate transformation of the allowed type.
In standard reading of the KK theory, the constant q in V.3 is interpreted as the ratio e-charge/mass. But in the geodesic equations V.4 the derivatives, denoted by the dot, are with respect to the total five dimensional arc length and this is, due to V.1, different from the 4-dimensional arc length $ds_4 = d\tau$ (the eigentime). Under the cyclic assumption with V.3 we have $ds^2 = d\tau^2 - q^2 ds^2 \;\Rightarrow ds^2 \cdot (1+q^2) = d\tau^2$ (see also [WL]). To compare V.4 with standard electromagnetic theory, we have to replace in it the derivative with respect to the arc-length s , with the derivative with respect to the 4-D arc length τ ($ds \to d\tau$). Doing this, the structure of equation V.4 keeps the same, if we define the dot as the derivative with respect to τ and if we replace $q \to q/\sqrt{(1+q^2)}$, which now has to be interpreted as the ratio e-charge/mass.
The "covariance"-problem of the theory arises from the 5'th dimension and the interpretation of A_μ as components of a four vector. So one should try to find an embedding of the 4-D space into the 5-D Kaluza-Klein space, with the desired properties, that is, in the optimal case, with the same geodesic equations. The development and analysis of a satisfying, "covariant" Kaluza-Klein Theory, which considers a "full dependency" on all dimensional coordinates, is a special field of research (see e.g [WL, Le]) . Such an ansatz results in generally "orthogonal forces" (orthogonal to the 4-D hyperplane), looses gauge invariance, and results also in some other interesting phenomena (see [Le] and literature cited there). [Le] constructs a theory in which the forces are again gauge invariant, but it would not be astonishing (for me), if in a "final" theory, gauge invariance must be dropped. As seen, "using some gauge" has in this theory something to do with choosing a coordinate system, but in general relativity a lot of other, "classically conserved" quantities depend on the coordinate system. Noether's second theorem does not allow conserved tensor components in general relativity [BB,FFM,No]. I not want to discuss and go deeper inside the research about KK-Theory. A common effect of all these approaches is also that electromagnetic and gravitation forces are not independent, even though the authors do not mention this explicitly.

Geodesics from a KK theory– like ansatz

The model discussed in this article seems to not fit together with the standard KK-Theory. Nevertheless it may be of interest to see how it looks under a similar point of view.
For this let's come back to the stationary metric II.6 . With m^2 as defined on the right side of equation II.9 , but now keeping β as another independent coordinate, the metric is

$$ds^2 = a^2 dt^2 - b^{-2} dr^2 - r^2 d\Omega + m^2 dr^2 - d\beta^2 \tag{V.5}$$

and the associated Lagrange function, written in the usual way,

$$L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + (m\dot{r})^2 - \dot{\beta}^2 \quad .$$

Note, the metric $g_{\mu\nu}$ is here already the target metric II.8 in section II .
Let now $\dot{\beta} = B_\mu \dot{x}^\mu + B_4 \dot{y}$. In the following we use Latin indexes to run from 0 to 4 and with the additional definitions

$$\begin{aligned} &B_j := \beta_{,j} \; (\; j=0\ldots4),\, B := B_j \dot{x}^j \equiv \dot{\beta} \\ &P_j := \delta_{rj} m,\; P := P_j \dot{x}^j = m\dot{r} \end{aligned}$$

(an index "r", e.g. B_r , denotes the index corresponding to a radial quantity and so on), the Lagrangian simplifies to

$$L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - B^2 + P^2 \tag{V.6}$$

For an expression $X = X_j \dot{x}^j$ we have $\partial X / \partial \dot{x}^j = X_j$ and $\partial X / \partial x^j = X_{,j}$, so

$$\frac{d}{ds}\frac{\partial X}{\partial \dot{x}^j} - \frac{\partial X}{\partial x^j} = \dot{X}_j - X_{,j}$$

and the Euler-Lagrange equations are

$$\gamma_j = B(\dot{B}_j - B_{,j}) + \dot{B}B_j + PP_{,j} - \frac{d}{ds}(PP_j),\ \ \gamma_j := Du_j \ \text{ for } \ j < 4, \gamma_4 = 0$$

(with Du_j as defined in V.4) . Because of $B_{j,k} = B_{k,j} \Rightarrow \dot{B}_j = B_{,j}$, in the geodesic equations above no electromagnetic field will appear. The field is, as also seen obviously in the definition, a pure gauge field. With the standard KK-Ansatz $B_{\mu} = \Phi A_{\mu}, \ B_4 = \Phi$, this last symmetry reads as $\Phi(A_{j,k} - A_{k,j}) + (\Phi_{,k} A_j - \Phi_{,j} A_k) = 0$, and this may be interpreted that the field tensor ($F_{jk} = A_{j,k} - A_{k,j}$) annihilates with the second summand. The geodesic equations now reduce to

$$\gamma_j = \dot{B}B_j + PP_{,j} - \frac{d}{ds}(PP_j) \quad .$$

For the derivatives of P one calculates

$$PP_{,j} = \frac{(w^2)_j}{2} = \frac{(\dot{r})^2}{2}(m^2)_{,j} = \dot{r}^2 m m' \delta_{rj},\ \ \frac{d}{ds}(PP_j) = \frac{d}{ds} m^2 \dot{r} \cdot \delta_{rj} .$$

and so for the Euler-Lagrange equations

$$\gamma_j = B_j \dot{B} + (\dot{r}^2 m m' - \frac{d}{ds} m^2 \dot{r})\delta_{rj} = -m(\frac{d}{ds} m\dot{r})\delta_{rj}$$

or

$$\begin{aligned} \ddot{x}^{\sigma} + \Gamma^{\sigma}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} &= B^{\sigma}\dot{B} - m(\frac{d}{ds} m\dot{r}) g^{r\sigma} \\ \text{and} \qquad 0 &= B_4 \dot{B} \end{aligned} \tag{V.7}$$

The last equation has the solutions $B_4 = 0$ and $\dot{B} = 0$ (or both vanishing). The solution with $B_4 = 0, \ \dot{B} \neq 0$ splits itself into a set of different types of geodesics, and specifically contains , with $B_j = 0$ for $j \neq r$ and $B_r = m.$, the geodesics of the target metric. In this last case, β may be interpreted as a gauge field, which forces trajectories to stay in the submanifold, defined by $d\beta = m dr$ or, e.g. if the target metric is the flat one, the field β "keeps the space flat". We will discuss other possible geodesics of type $B_4 = 0$ below. While for $B_4 = 0$ the path in the y-dimension is arbitrary, that means this dimension is artificial and superfluous, this is not the case for the solutions $B = \dot{\beta} = const.$ and. Submanifolds, created by sets of geodesics are by construction geodesic submanifolds. The manifolds we discussed in section III are special examples of this.
We end this section with one final remark.

Where in the approach above, the electromagnetic field tensor always vanishes (pure gauge fields), the complex metric IV.6 offers the possibility to define a non-vanishing one in the metric via

$$\begin{gathered}\langle A\,, dx\rangle = i\langle u\,, du\rangle \Rightarrow A_\mu = i\cdot u_\nu \frac{\partial \bar{u}^\nu}{\partial x^\mu} \\ ds^2 = \langle u\,, u\rangle + \langle A\,, dx\rangle^2\end{gathered} \quad .$$

With the parametrization of (IV.7) one gets for the field

$$\begin{gathered}u_0 \frac{\partial \bar{u}^0}{\partial x^\mu} = x_\mu - \delta^0_\mu x_0 - ir\,\omega_{,\mu}\,,\quad \sum_{j=1}^{3} u_j \frac{\partial \bar{u}^j}{\partial x^\mu} = x_\mu - \delta^0_\mu x_0 - ir\,\eta_{,\mu} \\ \Rightarrow A_\mu = r\cdot(\omega-\eta)_{,\mu} \Rightarrow \langle A\,, dx\rangle = r\cdot(\omega-\eta)_{,\mu}\, dx^\mu = d\left(r\cdot(\omega-\eta)\right) - (\omega-\eta)\,dr\end{gathered} \quad ,$$

so this is a radial field, modulo a gauge field.

VI Discussion and Summary

We offered an embedding of space-time in a 10 dimensional De-Sitter space, composed of an 8-D light cone and a Kaluza-Klein S^1 Sphere. The light cone itself consists of two copies of the 4-D space, with congruent spatial angles. The diameter of the KK-Sphere determines the curvature of the total space and defines a minimal space resolution for the embedding. From this it is deduced that the diameter of the KK-dimension is of the size of a Planck length.

In the following , we extended the metric, by modifying the phase of the KK-dimension. This results, after diagonalization, in a non-stationary metric and a PDE-system. We shortly presented an associated simple particle picture for this equations. From general considerations we selected some special initial values and discussed the characteristic equations and the limit values for the metric. In this metric the gravitation "constant "and the speed of light is time and space dependent. This effects leads to a redshift of radiation from far objects as also to larger attractive forces on orbits and so describes main effects concerned with "dark energy" and "dark matter". At initial time, the metric is pure three dimensional, but may be seen as Schwarzschild-space, where the time dimension is joined to the spatial ones. For any $t>0$ the time becomes independent and this deformation decays with time and leads finally to the Schwarzschild space. But for any finite time at radial infinity, the spatial sub space converges to just a 2-dimensional sphere.

In section IV the space is considered in the context of basic complex projective geometry. This section is thought of as proposal for scientists with a deeper knowledge in this subject to take a closer look at this kind of space for further research to include quantum effects. The theory of Lagrangian submanifolds, for example, may lead to Dirac-like field equations as shown in [Ai] for C^2 . Ten dimensional spaces are also used in the Grand Unified Theory (the Georgi-Glashow model) and in some string theories. There is a large amount of literature (in mathematics and physics) about the standard complex hyperbolic space (e.g. in AdS/CFT correspondence ...), while physics literature concerning the complex de-Sitter space seems to be very sparse.

The last section V examines the proposed embedding in the context of classical Kaluza-Klein theory. Whereas the electromagnetic theory could not be inferred from this embedding, the set of geodesic submanifolds leads to spaces, which we considered in section III .

Appendix A (Some properties of classical dS and AdS)

A general, isotropic, stationary metric is written in the form

$$ds^2 = a^2(r)\,dt^2 - b^{-2}(r)\,dr^2 - r^2\, d\,\Omega \qquad \text{(A1)}$$

with $a^2(r) = 1+U(r),\quad b^2(r) = 1+V(r)$.

For $U=V>0$ set $U=\sinh^2(\eta)$ and for $U=V<0$ set $U=-\sin^2(\eta)$ the metric than is:

$$ds^2 = \cosh^2(\eta)\cdot dt^2 - d\,\eta^2 - r^2(\eta)\; d\,\Omega, \qquad \text{(A2)}$$

resp. $$ds^2 = \cos^2(\eta)\cdot dt^2 - d\,\eta^2 - r^2(\eta)\; d\,\Omega$$

A Friedman-Lemaitre-Robertson-Walker (FLRW) has the form:

$$ds^2 = dt^2 - K^2(t)\cdot\left(d\,\eta^2 - r^2(\eta)\; d\,\Omega\right) \qquad \text{(A3)}$$

Characteristic properties of **dS**:

– Hypersurface in $\boldsymbol{R}^{1,4}$: $x^2 \equiv \langle x\,,x\rangle \equiv x_0^2 - \sum_1^4 x_i^2 = -\alpha^2$

– Metric of $\boldsymbol{R}^{1,4}$: $ds^2 = \mathrm{d}\,x_0^2 - \sum_1^4 \mathrm{d}\,x_i^2$

– dS Metric (stationary): $U(r)=V(r)=-(r/\alpha)^2$, $r=\alpha\cdot\sin(\eta)$

– in FLRW coordinates $r=\sin(\eta),\, K=\alpha\cdot\cosh(t)$

– Newton potential: $U_N = \dfrac{U}{2} = \dfrac{-(r/\alpha)^2}{2}$

– Ricci curvature + Einstein tensor: $Ric = 3\cdot\alpha^{-2}\cdot g\,,\ \Rightarrow G=-Ric=-3\cdot\alpha^{-2}$

– Einstein equation: $G=-\Lambda\cdot g \Rightarrow \Lambda = 3\cdot\alpha^{-2}\cdot g > 0$

Characteristic properties of **AdS**:

– Hypersurface in $\boldsymbol{R}^{2,3}$: $x^2 \equiv \langle x\,,x\rangle \equiv x_0^2 + x_1^2 - \sum_2^4 x_i^2 = \alpha^2$

– Metric in $\boldsymbol{R}^{2,3}$: $ds^2 = \mathrm{d}\,x_0^2 + \mathrm{d}\,x_1^2 - \sum_2^4 \mathrm{d}\,x_i^2$

– AdS Metric (stationary): $U(r)=V(r)=+(r/\alpha)^2$, $r=\alpha\cdot\sinh(\eta)$

- in FLRW coordinates $r = \sinh(\eta), K = \alpha \cdot \cos(t)$
- Newton potential: $U_N \quad = + \frac{1}{2} \cdot (\frac{r}{\alpha})^2$
- Ricci curvature + Einstein tensor : $Ric = -3 \cdot \alpha^{-2} \cdot g, \ \Rightarrow G = -Ric = 3 \cdot \alpha^{-2}$
- Einstein tensor : $G = -\Lambda \cdot g \Rightarrow \Lambda = -3 \cdot \alpha^{-2} \cdot g < 0$

Appendix B (Parametrization of dS and AdS in the 6 -D light cone)

In the following have a look how a global parametrization of **K**

$$\mathbf{K} = \{ x \in \boldsymbol{R}^{2,4} : x^2 := \langle x, x \rangle = 0 \}$$

influences the induced metrics.

Using spherical coordinates in the last 3 space-like coordinates (which now are treated as the usual space dimensions !), the metric of $\boldsymbol{R}^{2,4}$ reads as:

$$ds^2 = dx_0^2 + dx_1^2 - dx_5^2 - dr^2 - r^2 \, \mathrm{d}\Omega \ ,$$

where $\mathrm{d}\Omega$ is the usual 3-d surface element and **K** is defined through the equation

$$x_0^2 + x_1^2 = x_5^2 + r^2 . \qquad \text{(B1)}$$

1.) In circular parametrization,

$$x_0 = a \cdot \sin(\omega), x_1 = a \cdot \cos(\omega), x_5 = a \cdot \cos(\phi), r = a \cdot \sin(\phi), \ \ a := \cosh(\eta)$$

the metric on **K** becomes

$$ds^2 = \cosh^2(\eta) \cdot (d\omega^2 - d\phi^2 - \sin^2(\phi) d\Omega) \ . \qquad \text{(B2)}$$

Now **dS** is the section $x_1 = \cosh(\eta) \cdot \cos(\omega) = 1$. Putting this into the (B2) leads to the Friedman-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = d\eta^2 - \cosh^2(\eta)(d\phi^2 + \sin^2(\phi) d\Omega) \ .$$

with η as the eigentime parameter and radial parameter $r = \sin(\phi)$ (note this parameter r now is not the same as the “original” one).

AdS is the section $x_5 = \cosh(\eta) \cdot \cos(\phi) = 1$. Applying this on (B2) induces the stationary Anti-de-Sitter metric

$$ds^2 = \cosh^2(\eta) \cdot d\omega^2 - d\eta^2 - \sinh^2(\eta) d\Omega \ \cdot$$

with ω as the eigentime parameter and radial parameter $r = \sinh(\phi)$.

2.) In hyperbolic parametrization,

$$x_0 = a \cdot \sinh(\omega), x_1 = a \cdot \cosh(\phi), x_5 = a \cdot \cosh(\omega), r = a \cdot \sinh(\phi), \ \ a := \cos(\eta)$$

the metric on **K** is

$$ds^2 = \cos^2(\eta)\cdot(d\,\omega^2 - d\,\phi^2 - \sinh^2(\phi)\,d\,\Omega) \qquad \text{(B3)}$$

dS is the section $x_1 = \cos(\eta)\cdot\cosh(\phi) = 1$. Now, putting this into (B3), we get the stationary metric

$$ds^2 = \cos^2(\eta)\cdot d\,\omega^2 - d\,\eta^2 - \sin^2(\eta)\,d\,\Omega$$

with ω as the eigentime and radial parameter $r = \sin(\eta)$.

AdS is the section $x_5 = \cosh(\omega)\cdot\cos(\eta) = 1$. Again applying this relation to (B3), leads to the FLWR metric

$$ds^2 = d\,\eta^2 - \cos^2(\eta)(d\,\phi^2 + \sinh^2(\phi)\,d\,\Omega)$$

with η as the eigentime parameter and radial parameter $r = \sinh(\phi)$ (again this parameter r now is not the same as the "original" one).

Appendix C (8-D embedding)

On $R^{2,6}$ we start with the same parametrization as for $R^{2,8}$, but without the Kaluza-Klein sphere, and we do not restrict the space on a sphere. The metric is equivalent to (II.5)

$$ds^2 = A^2\,d\,\omega^2 + dA^2 - dr^2 - r^2\,d\,\phi^2 - r^2\,d\,\Omega$$

but with no restriction for A. So set $\omega = t$ and $A = 1 + V(r)$ to get the metric of the form

$$ds^2 = A^2\,d\,\omega^2 - A^{-2}\,dr^2 - r^2\,d\,\Omega$$

we have to integrate just

$$r^2\,d\,\phi^2 = dA^2 - \frac{V}{1+V}\,dr^2 = \left(-V + \left(\frac{1}{2}\frac{dV}{dr}\right)^2\right)\cdot\frac{dr^2}{1+V} \quad .$$

This is always possible, for the considered potentials $V = -r^2$ (**dS**), $V = +r^2$ (**AdS**) and $V = -r_0/r$ (**Sch).**

Appendix D (The metric of complex de-Sitter space)

In the following I point out the "motivation" for the projective metric in IV.

Let $\langle .,. \rangle$ be the scalar product on $C^{d,n+1-d}$. On $M_h = \{z : \langle z, z\rangle = h\}$, $h \in \mathbb{R}$, define projective coordinates $u_j = \frac{z_j}{z_n}$, $j = 0\ldots n$, with u_n = 1 (to keep this last constant coordinate is not usual but simplifies the notation in the following) .
The orthogonal component of a vector at z is an element of the kernel of the map du , that is, vectors parallel to the coordinate "vector" z. The Cartesian coordinates $v: \{v_j,\ j=\ldots n\}$ of a vector in $C^{d,n+1-d}$ could be expressed through the coordinates $\hat{v}$ in the basis ∂_{u_i} (extended to n+1 dimensions) and the orthogonal vector on M_k

$$v = z_n \hat{v} + v_n u \text{ with } \hat{v}_n = 0 \qquad \text{(C1).}$$

We also have the usual decomposition of a vector in a tangential and orthogonal part $v = v^T + h^{-1} \cdot \langle v, z \rangle, \; z := v^T + b_v \cdot z$. Now define the induced metric on M to be just

$$(\hat{v}, \hat{w}) = \langle v^T, w^T \rangle \quad .$$

The left side is just an n-dimensional metric, the n-th components are zero (see C1)! Now inserting (C1) into $\langle v^T, w^T \rangle = \langle v, w \rangle - h \cdot b_v \bar{b}_w$ and using $|z_n|^2 \cdot \langle u, u \rangle = h$ and $\langle v, u \rangle = z_n \langle \hat{v}, u \rangle + v_n \langle u, u \rangle$ leads finally after some lines of calculations to

$$(\hat{v}, \hat{w}) = |z_n|^2 \left(\langle \hat{v}, \hat{w} \rangle + |z_n|^2 \langle \hat{v}, u \rangle \langle u, \hat{w} \rangle \right) \quad .$$

Now we can drop the n-th component on the right side also. The final expression we obtain now $\langle u, u \rangle = \langle u, u \rangle_n + g_{nn}$, where $\langle u, u \rangle_n$ denotes also the reduced n-dimensional metric and $g_{nn} = \langle \partial_{z_n}, \partial_{z_n} \rangle$ is its last diagonal entry. For $C^{1,4}$, $g_{nn} = -1$ and with $h = -1$, $|z_n|^2 = \dfrac{1}{(1 - \langle u, u \rangle_n)}$, as in (IV.1).

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